Addendum to “Some General Formulas in the Sato Theory”

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In a recent short note1) we have obtained some general formulas of the Sato theory related to two pseudo-differential operators. In this addendum, following the procedure of ref. 1, we derive general formulas for the hierarchy of the two-dimensional Toda lattice (2DTL).

The linear problems related to the 2DTL hierarchy are2)

\[ L \phi_n = \lambda \phi_n, \quad (\phi_n)_t = B^{(m)}_n \phi_n, \quad (\phi_n)_x = C^{(m)}_n \phi_n \]
and

\[ M \psi_n = \frac{1}{2} \psi_n, \quad (\psi_n)_t = B^{(m)}_n \psi_n, \quad (\psi_n)_x = C^{(m)}_n \psi_n \]

where pseudo-difference operators \( L \) and \( M \) are defined by

\[ L = E + v_{1,n} + v_{2,n} E^{-1} + v_{3,n} E^{-2} + \cdots, \]
\[ M = v_{0,n} E^{-1} + v_{1,n} + v_{2,n} E + v_{3,n} E^2 + \cdots; \]

\( E \) is a shift operator; \( u_{1,n} \) and \( v_{1,n} \) are the functions of the variables \( n, t_1, t_2, \ldots, y_1, y_2, \ldots \). The operators \( B^{(m)}_n = (L^m)_{\geq 0} \) and \( C^{(m)}_n = (M^m)_{\leq -1} \), where \((\cdot)_{\geq k}\) means that part of the shift operator containing \( E^k \) \((j \geq k)\), and \((\cdot)_{\leq k}\) means that part containing \( E^{-k} \) \((j \leq k)\). The 2DTL hierarchy is3) derived from the following four Lax equations related to eqs. (1) and (2):

\[ L \phi_n = [B^{(m)}_n, L], \]
\[ L \psi_n = [C^{(m)}_n, L], \]
\[ M \phi_n = [B^{(m)}_n, M], \]
\[ M \psi_n = [C^{(m)}_n, M]. \]

Let us first consider a general formula for \( B^{(m)}_n \). Set

\[ B^{(m)}_n = E^{m} + \sum_{j=1}^{m} b^{(m)}_{j,n} E^{-j} = (L^m)_{\geq 0}, \quad (m = 1, 2, \ldots). \]

Similar to the previous paper,1) the coefficients of \( B^{(m)}_n \) are determined from the equation

\[ [B^{(m)}_n, L]_{\geq 1} = (B^{(m)}_n L - LB^{(m)}_n)_{\geq 1} = 0. \]

The general formulas for \( b^{(m)}_{j,n} \) are given by

\[ b^{(m)}_{j,n} = \sum_{l=1}^{m} u_{j+1,n+l-1}, \]

For the operator \( C^{(m)}_n \), we write

\[ C^{(m)}_n = \sum_{j=0}^{m-1} c^{(m)}_{j,n} E^{-m}, \]

and obviously

\[ c^{(m)}_{0,n} = v^{(m)}_n = \prod_{j=1}^{m} v_{0,n-j+1}. \]

Then, noticing that \( [C^{(m)}_n, M]_{\leq -2} = 0 \) and with the help of setting

\[ c^{(m)}_{j,n} = v^{(m-j)}_n - (j = 1, 2, \ldots, m - 1), \]

we can reach the following general formulas for \( c^{(m)}_{j,n} \):

\[ c^{(m)}_{j,n} = v^{(m-j)}_n \left( \sum_{l=0}^{m-j} \left( \frac{1}{n} \sum_{i=1}^{l} \left( c^{(m)}_{j+i,n-l-i} \right) \right) \right), \quad (j = 1, 2, \ldots, m - 2). \]

With these formulas in hand, we further work out the explicit forms of the Lax equations (5a)–(5d). For (5a), we have

\[ (u_{j+1,n})_{n} = u_{m+j+1,n+m} - u_{m+j+1,n}, \]
\[ + \sum_{l=1}^{m} b^{(m)}_{j,n} u_{m+j-l+1,n+l-1}, \quad (j = 0, 1, \ldots). \]

For (5b), we have

\[ (u_{j+1,n})_{n} = e^{(m)}_{j+1,n} - e^{(m)}_{j+1,n+1}, \]
\[ (u_{j+1,n})_{n} = e^{(m)}_{j+1,n} - e^{(m)}_{j+1,n+1}, \]
\[ + \sum_{l=0}^{m-j-1} c^{(m)}_{j+1,n-l} u_{m+j-l+1,n-l} - c^{(m)}_{j+1,n-l} u_{m+j-l+1,n}, \quad (j = 1, 2, \ldots, m - 1), \]
\[ (u_{j+1,n})_{n} = \sum_{l=0}^{m-j-1} c^{(m)}_{j+1,n-l} u_{m+j-l+1,n-l} - c^{(m)}_{j+1,n-l} u_{m+j-l+1,n}, \quad (j = m, m+1, \ldots). \]
Thus the general representation of the hierarchy of 2DTL is obtained. These formulas are explicit and simple. They are also closely related to the Blaszak–Marciniak lattices.3,4) This project is supported by the National Science Foundation of China.