The $N$-soliton solutions of the sine-Gordon equation with self-consistent sources

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Abstract

The hierarchy of the sine-Gordon equation with self-consistent sources is derived by using the eigenfunctions of recursion operator. The bilinear form of the sine-Gordon equation with self-consistent sources is given and the $N$-soliton solutions are obtained through Hirota method and Wronskian technique, respectively. Some novel determinantal identities are presented to treat the nonlinear term in the time evolution and finish the Wronskian verifications.

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1. Introduction

The study of the soliton equations with self-consistent sources (SESCS) has received considerable attention in recent years [1–14]. The SESCS, describing important physical processes such as the interactions between different waves [2,3], can be constructed through some mathematical ways [4–9]. One of simple methods is using the high-order constrained flows of soliton equations, namely, the high-order constrained flows of soliton equations are considered as the stationary equations of the SESCS [7–9]. Most of the sources obtained in this way are related to eigenfunctions because the variational derivatives of eigenvalues are related to eigenfunctions. Some studies have also shown that the SESCS exhibit multisoliton solutions [7,10–14]. With the help of some special treatments, the inverse scattering method and Darboux transformation have been successfully used to find $N$-soliton solutions of the SESCS such as the KdV, AKNS,
modified KdV, nonlinear Schrödinger and Kaup–Newell hierarchies with self-consistent sources. It will be shown from the solutions that the sources may result in the variation of the velocity of solitons [5,15].

One of the purposes of this paper is to derive the hierarchy of the sine-Gordon equation with self-consistent sources in a general way which is directly based on the eigenfunctions of recursion operator. This method, different slightly from the one using constrained flows, is easy to give the Lax representations of the hierarchy. On the other hand, we also hope to find the multisoliton solutions of the sine-Gordon equation self-consistent sources (SGESCS) through Hirota method [16] and Wronskian technique [17–20]. These two direct methods both depend on the bilinear forms of the evolution equations. Hirota method provides a remarkably simpler technique for obtaining the $N$-soliton solutions in the form of an $N$th-order polynomial in $N$ exponentials. Wronskian technique provides an alternative formulation of the $N$-soliton solutions, in terms of some function of the Wronski determinant of $N$ functions, which allows verification of the solutions by direct substitution because differentiation of a Wronskian is easy and its derivatives take similar compact forms. The basic idea of our obtaining the exact $N$-soliton solutions in this paper are as follows. We first present a set of dependent variable transformations to write out the bilinear form of the SGESCS by which we can derive one-, two-, even three-soliton solutions successively through the standard Hirota’s approach. These results can help us to find out the time evolution easily and conjecture a general formula which denotes $N$-soliton solution but is only conjectured and not verified. Next, with the help of the informations on the time evolution obtained by means of Hirota method, we can construct Wronskians and try to verify it to satisfy the related bilinear equations. Due to a nonlinear term (led to by the concerned source) in the time evolution, some derivatives of a Wronskian do not possess the compact forms. We have to develop some novel determinantal identities and employ some special treatments which are different from the known standard Wronskian technique [17–20] so that we can finish the Wronskian verifications. We also present a process to show that the solutions of the bilinear equations obtained through the above two direct methods are the same for recovering the solutions of SGESCS. To our knowledge, it is the first time to obtain the SGESCS and solve it by Hirota method and Wronskian technique.

The paper is arranged as follows. We first derive the hierarchy of the SGESCS in Section 2. Then we give the bilinear form of the SGESCS and solve it by means of Hirota method and Wronskian technique in Sections 3 and 4, respectively.

2. The hierarchy of the SGESCS

In this section, we derive the hierarchy of the SGESCS. Consider the following $2 \times 2$ eigenvalue problem [21]

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}_x = M \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}, \quad M = \begin{pmatrix}
-\lambda & \frac{u_x}{2} \\
-\frac{u_x}{2} & \lambda
\end{pmatrix}
\] (2.1a)
and the general time dependence
\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}_t = N
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},
\]
(2.1b)
where \( u = u(t,x) \) is a potential function satisfying \( u \to 0 \) as \( |x| \to \infty \), \( A, B, C \) are functions independent of \( \varphi_1 \) and \( \varphi_2 \). The related compatibility condition, zero curvature equation, is
\[
M_t - N_x = [N,M],
\]
(2.2)
which can be further reformulated by
\[
A = \frac{1}{2} \partial^{-1} u_x (B + C) + A_0,
\]
(2.3a)
\[
\frac{1}{2} \begin{pmatrix} u_x \\ -u_x \end{pmatrix}_t = L \begin{pmatrix} -B \\ C \end{pmatrix} - 2\lambda \begin{pmatrix} -B \\ C \end{pmatrix} + u_x A_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
(2.3b)
where
\[
L = \begin{pmatrix}
-\partial - \frac{1}{2} u_x \partial^{-1} u_x & \frac{1}{2} u_x \partial^{-1} u_x \\
-\frac{1}{2} u_x \partial^{-1} u_x & \partial + \frac{1}{2} u_x \partial^{-1} u_x
\end{pmatrix},
\]
(2.4)
\( \partial = \partial/\partial x, \partial^{-1} = \int_{-\infty}^{x} dx \) and \( A_0 = A|_{u=0} \). Unlike the traditional treatments [21], we expand \((-B,C)^T\) as
\[
\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{l=1}^{m} \begin{pmatrix} -b_l \\ c_l \end{pmatrix} (2\lambda)^{l-m-1} + \frac{1}{2} \sum_{j=1}^{N} \begin{pmatrix} \frac{x_j}{\lambda + \lambda_j} + \frac{\beta_j}{\lambda - \lambda_j} \end{pmatrix},
\]
(2.5)
where \( \{\lambda_j\} \) are \( N \) distinct eigenvalues satisfying
\[
\begin{pmatrix}
\varphi_{1j} \\
\varphi_{2j}
\end{pmatrix}_x = \begin{pmatrix} -\lambda_j & u_x \\ -u_x & \lambda_j \end{pmatrix} \begin{pmatrix}
\varphi_{1j} \\
\varphi_{2j}
\end{pmatrix}, \quad (j = 1,2,\ldots,N).
\]
Substituting (2.5) into (2.3b) and setting \( A_0 = -\frac{1}{2}(2\lambda)^{-m} \) yield
\[
\frac{1}{2} \begin{pmatrix} u_x \\ -u_x \end{pmatrix}_t = \sum_{l=1}^{m} L \begin{pmatrix} -b_l \\ c_l \end{pmatrix} (2\lambda)^{l-m-1} + \frac{1}{2} \sum_{j=1}^{N} L \begin{pmatrix} \frac{x_j}{\lambda + \lambda_j} + \frac{\beta_j}{\lambda - \lambda_j} \end{pmatrix}
\]
\[
- \sum_{l=1}^{m} \begin{pmatrix} -b_l \\ c_l \end{pmatrix} (2\lambda)^{l-m} - \sum_{j=1}^{N} (x_j + \beta_j)
\]
\[
+ \sum_{j=1}^{N} \frac{\lambda_j x_j}{\lambda + \lambda_j} - \frac{\lambda_j \beta_j}{\lambda - \lambda_j} - \frac{u_x}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}(2\lambda)^{-m}.
\]
(2.6)
Then, comparing the coefficients of the powers of $2\lambda$ and $1/(\lambda \pm \lambda_j)$ in Eq. (2.6), we have

$$
\frac{1}{2} \left( \begin{array}{c}
u_x \\
-u_x
\end{array} \right)_t = -\left( \begin{array}{c}-b_m \\
c_m
\end{array} \right) - \sum_{j=1}^{N} (\alpha_j + \beta_j),
$$

$$
\left( \begin{array}{c}-b_l \\
c_l
\end{array} \right) = \frac{1}{2} L^{-l} \left( \begin{array}{c}u_x \\
u_x
\end{array} \right), \quad (l = 1, 2, \ldots, m),
$$

$$
L\alpha_j = -2\lambda_j x_j, \quad L\beta_j = 2\lambda_j \beta_j, \quad (j = 1, 2, \ldots, N)
$$
or further

$$
\left( \begin{array}{c}u_x \\
-u_x
\end{array} \right)_t = -L^{-m} \left( \begin{array}{c}u_x \\
u_x
\end{array} \right) - 2 \sum_{j=1}^{N} (\alpha_j + \beta_j)
$$

$$
= L^{-(m+1)} \left( \begin{array}{c}u_{xx} \\
-u_{xx}
\end{array} \right) - 2 \sum_{j=1}^{N} (\alpha_j + \beta_j), \quad (2.7a)
$$

$$
L\alpha_j = -2\lambda_j x_j, \quad L\beta_j = 2\lambda_j \beta_j, \quad (j = 1, 2, \ldots, N). \quad (2.7b)
$$

If we only consider the case of $m$ being an odd number ($m = 2h - 1, \ h = 1, 2, \ldots$) and take

$$
\alpha_j = -2\lambda_j (\varphi_{2j}^2, \varphi_{1j}^2)^T, \quad \beta_j = 2\lambda_j (\varphi_{1j}, \varphi_{2j})^T, \quad (j = 1, 2, \ldots, N),
$$

we can reach the hierarchy of the SGESCS which can be described as

$$
u_{xt} = F^{-h}u_{xx} + 2 \sum_{j=1}^{N} (\varphi_{1j}^2 + \varphi_{2j}^2)_x, \quad (h = 1, 2, \ldots), \quad (2.8a)
$$

$$
\varphi_{1j,x} = -\dot{\lambda}_j \varphi_{1j} + \frac{u_x}{2} \varphi_{2j}, \quad (2.8b)
$$

$$
\varphi_{2j,x} = -\frac{u_x}{2} \varphi_{1j} + \dot{\lambda}_j \varphi_{2j}, \quad (2.8c)
$$

where $\{(\varphi_{1j}^2 + \varphi_{2j}^2)_x\}$ are $N$ self-consistent sources, the operator

$$
F = \delta^2 + u_x^2 + u_{xx} \delta^{-1} u_x \quad (2.9)
$$

and

$$
F^{-1} = \cos u \delta^{-1} \cos u \delta^{-1} + \sin u \delta^{-1} \sin u \delta^{-1}. \quad (2.10)
$$

Obviously, it is easy to obtain the Lax representations of the hierarchy (2.8).

Especially, taking $h=1$ in hierarchy (2.8), we can obtain the first equation as follows:

$$
u_{xt} = \sin u + 2 \sum_{j=1}^{N} (\varphi_{1j}^2 + \varphi_{2j}^2)_x, \quad (2.11a)$$
\[ \varphi_{1j,x} = -\lambda_j \varphi_{1j} + \frac{u_x}{2} \varphi_{2j} , \] (2.11b)

\[ \varphi_{2j,x} = -\frac{u_x}{2} \varphi_{1j} + \lambda_j \varphi_{2j} , \] (2.11c)

which we call the SGESCS.

3. Solving the SGESCS by Hirota method

Now, we solve Eq. (2.11) by means of Hirota method. Physically, the number of solitons is determined by the simple poles of the scattering data 1/a(\lambda) [21,22]. On the other hand, it has been shown in the above section that the \( N \) distinct eigenvalues \( \{\lambda_j\} \) just lead to \( N \) self-consistent sources. That is to say, Eq. (2.11), the sine-Gordon equation with \( N \) self-consistent sources, only admits \( N \)-soliton solution.

With the help of the following dependent variable transformation:

\[ u = 2i \ln \frac{\tilde{f}}{f}, \] (3.1a)

\[ \varphi_{1j} = \frac{\tilde{g}_j}{f} + \frac{g_j}{f}, \quad \varphi_{2j} = i \left( \frac{\tilde{g}_j}{f} - \frac{g_j}{f} \right), \quad (j = 1, 2, \ldots, N), \] (3.1b)

the SGESCS (2.11) can be transformed into the bilinear form

\[ D_x D_t f \cdot \tilde{f} = \frac{1}{2} \left( f^2 - \tilde{f}^2 \right) - 8i \sum_{j=1}^{N} \lambda_j g_j^2 , \] (3.2a)

\[ D_x \tilde{g}_j \cdot f = -\lambda_j g_j \tilde{f}, \quad \lambda_j \in \mathbb{R}, \quad (j = 1, 2, \ldots, N), \] (3.2b)

where \( \tilde{f} \) and \( \tilde{g}_j \) are the complex conjugates of \( f \) and \( g_j \), and \( D \) is the well-known Hirota bilinear operator

\[ D^n_x D^n_t a \cdot b = (\partial_x - \partial_{x'})^n (\partial_t - \partial_{t'})^n a(x,t)b(x',t') \big|_{x'=x,t'=t} . \]

To derive \( N \)-soliton solution, we expand

\[ f(x,t) = 1 + f^{(2)} e^2 + f^{(4)} e^4 + \cdots , \] (3.3a)

\[ g_j(x,t) = g_{j}^{(1)} e + g_{j}^{(3)} e^3 + \cdots , \quad (j = 1, 2, \ldots, N) \] (3.3b)

and take

\[ g_{j}^{(1)} = \sqrt{\beta_j(t)} e^{\xi_j} , \]

\[ \xi_j = k_j x + \omega_j t + \int_{0}^{t} \beta_j(z) \, dz + \xi_{j}^{(0)} , \quad (j = 1, 2, \ldots, N) , \] (3.4)

where \( k_j, \omega_j, \xi_{j}^{(0)} \) are all real constants and \( \beta_j(t) \) is an arbitrary real function of \( t \). Then through the standard Hirota’s process [16,21], it is easy to find 1- and
2-soliton solutions which can be denoted, respectively, by
\[ f = 1 + ie^{2\xi_1}, \quad g_1 = \sqrt{\beta_1(t)} e^{\xi_1} \] (3.5)
and
\[ f = 1 + i(e^{2\xi_1} + e^{2\xi_2}) - \left(\frac{k_j - k_l}{k_j + k_l}\right)^2 e^{2\xi_1 + 2\xi_2}, \] (3.6a)
\[ g_1 = \sqrt{\beta_1(t)} e^{\xi_1} + i\frac{k_1 - k_2}{k_1 + k_2} \sqrt{\beta_1(t)} e^{\xi_1 + 2\xi_2}, \] (3.6b)
\[ g_2 = \sqrt{\beta_2(t)} e^{\xi_2} - i\frac{k_1 - k_2}{k_1 + k_2} \sqrt{\beta_2(t)} e^{2\xi_1 + \xi_2}, \] (3.6c)
where \( \omega_j = 1/4k_j \) and \( \lambda_j = -k_j \). We can further conjecture the \( N \)-soliton solution where
\[ f = \sum_{\mu=0,1} \exp\left\{ \sum_{j=1}^N \mu_j \left(2\xi_j + \frac{\pi}{2}\right) + \sum_{1 \leq j < l \leq N} \mu_j \mu_l A_{jl}\right\}, \] (3.7a)
\[ g_h = \sqrt{\beta_h(t)} \sum_{\mu=0,1} \exp\left\{ \mu_h \xi_h + \sum_{j=1}^{h-1} \mu_j \left(2\xi_j + \frac{1}{2} A_{jh} + \frac{\pi}{2}\right) + \sum_{j=h+1}^N \mu_j \left(2\xi_j + \frac{1}{2} A_{jh} - \frac{\pi}{2}\right) + \sum_{1 \leq j < l \leq N, j \neq h} \mu_j \mu_l A_{jl}\right\}, \] (3.7b)
\[ \omega_j = \frac{1}{4k_j}, \quad \lambda_j = -k_j, \quad e^{4\omega_j} = \left(\frac{k_j - k_l}{k_j + k_l}\right)^2, \] (3.7c)
where the sum over \( \mu = 0,1 \) refers to each of the \( \mu_j, (j = 1,2,\ldots,N) \) and \( \{k_j\} \) satisfy \( k_1 < k_2 < \cdots < k_N \). This conjecture will be shown to be right in Appendix A.

4. Solving the SGESCS by Wronskian technique

4.1. New properties of Wronskians

A Wronskian is a determinant of an \( N \times N \) matrix with columns \( \phi^{(0)}, \ldots, \phi^{(N-1)} \), where \( \phi^{(0)} = \phi = (\phi_1(x,t), \ldots, \phi_N(x,t))^T \) and \( \phi^{(j)} = \partial_t \phi^{(0)}/\partial x^j \), written
\[ W = |\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(N-1)}| = |0,1,\ldots,N-1| = |\overset{\rightarrow}{N-1}|. \] (4.1)
Here and below, we adopt a notation for the general determinant of the form [17]:
\[ |\overset{\rightarrow}{N-j}, N-k_1, N-k_2, \ldots, N-k_{j-1}|, \]
where \( \overset{\rightarrow}{N-j} \) denotes \( N-j+1 \) consecutive columns 0,1,2,\ldots,\( N-j \).
To finish the verification of the $N$-soliton solution, we should generalized some
determinantal properties. First, we give the following identity which is easily proved
by using the definition of a determinant.

**Proposition 1.** Let $|A| = |x_1, \ldots, x_N|$, $x_j = (a_{1j}, \ldots, a_{Nj})^T$ and $b x_j = (b_{1a_{1j}}, \ldots, b_{Na_{Nj}})^T$, we have

$$
\left( \sum_{1 \leq j_1 < j_2 < \cdots < j_N \leq N} b_{j_1} b_{j_2} \cdots b_{j_N} \right) |A| = \sum_{1 \leq j_1 < j_2 < \cdots < j_N \leq N} |x_1, \ldots, b x_{j_1}, \ldots, b x_{j_2}, \ldots, b x_{j_N}, \ldots, x_N| . \quad (4.2)
$$

With this in hand, it is not difficult to find the following property for an $(N-1) \times (N-1)$ Wronskian.

**Proposition 2.** Suppose that an $(N-1) \times (N-1)$ Wronskian $|N - 2|$ satisfies

$$
\phi_{j,xx} = k_j^2 \phi_j, \quad (j = 1, 2, \ldots, N-1) , \quad (4.3)
$$

set

$$
P(N-m) = \left( \sum_{1 \leq j_1 < j_2 < \cdots < j_{N-m} \leq N-1} k_{j_1}^2 k_{j_2}^2 \cdots k_{j_{N-m}}^2 \right) \left| \begin{array}{c} N - 2 \end{array} \right| = \sum_{0 \leq l_1 < l_2 < \cdots < l_{N-m} \leq N-2} |0, 1, \ldots, k^2 x_{l_1}, \ldots, k^2 x_{l_2}, \ldots, k^2 x_{l_{N-m}}, \ldots, N-2| , \quad (4.4)
$$

again, introduce the matrix

$$
M(h, j) = [0, 1, \ldots, h-1, h+1, h+2, \ldots, j-1, j+1, \ldots, N]_{(N-1) \times (N-1)} \quad (4.5)
$$

and in the special cases let

$$
M(0, N) = [1, 2, \ldots, N-2, N-1] , \quad (4.6a)
$$

$$
M(h, h+1) = [0, 1, \ldots, h-1, h+1, h+2, \ldots, N] , \quad (4.6b)
$$

then we have

$$
P(N-m) = (-1)^{m-1} \sum_{j=0}^{m-1} (-1)^j |M(j, 2m - j - 1)| , \quad (2 \leq m \leq \left[ \frac{N}{2} \right]) \quad (4.7)
$$

and

$$
P(N-m) = (-1)^{N-m-1} \sum_{j=-1}^{N-m-1} (-1)^j |M(2m - N + j, N - j - 1)| ,
$$

$$
\left( \left[ \frac{N}{2} \right] + 1 \leq m \leq N - 1 \right) . \quad (4.8)
$$
Obviously, when \( m = N - 1 \), (4.8) casts into
\[
\left| \sum_{j=1}^{N-1} k_j^2 \right| N-2 = -\left| \sum_{j=1}^{N-1} k_j^2 \right| N-2, N-1 + | N-2, N | \tag{4.9}
\]
or in \( N \times N \) case
\[
\left( \sum_{j=1}^{N} k_j^2 \right) | N-1 = -| N-2, N-1, N | + | N-2, N+1 | , \tag{4.10}
\]
which is a familiar and key formula in the previous references \[17–20\].

4.2. \( N \)-soliton solution in terms of the Wronskian form

In this part, we consider the direct verification of the \( N \)-soliton solution in Wronskian form for the SGESCS.

**Proposition 3.** Let
\[
\phi_j = e^{\pi i/4} [e^{\xi_j + \pi i/4} - (-1)^j e^{-\xi_j - \pi i/4}], \quad (j = 1, 2, \ldots, N) \tag{4.11}
\]
and \( \xi_j \) be described as the one in (3.4), then the Wronskians
\[
f = | N-1 | , \tag{4.12a}
\]
\[
g_h = (-1)^{h+N} \beta_h(t) \prod_{l=1}^{h-1} (k_h^2 - k_l^2) \prod_{l=h+1}^{N} (k_l^2 - k_h^2) | N-2, \tau_h | ,
\]
\[
\tau_h = (\delta_{h,1}, \ldots, \delta_{h,N})^T, \quad (h = 1, \ldots, N) \tag{4.12b}
\]
and \( \lambda_h = -k_h \) solve the bilinear equations (3.2), here we still set \( k_1 < k_2 < \cdots < k_N \).

**Proof.** The sine-Gordon equation in the bilinear form is [21]
\[
D_x D_t f \cdot f = \frac{1}{2} (f^2 - \tilde{f}^2) \tag{4.13}
\]
with the bilinear Bäcklund transformation [19,23]
\[
D_x \tilde{g} \cdot f = -\lambda g \tilde{f}, \quad D_t \tilde{g} f = -\frac{1}{4\lambda} g \tilde{f}, \quad \lambda \in \mathbb{R} , \tag{4.14}
\]
where the \( x \)-part is nothing but just (3.2b). Partly using the results for the sine-Gordon equation in Ref. [19], it is not difficult to finish the Wronskian verifications of (4.13) and (4.14) under the choice
\[
f = | N-1 | , \quad g = | N-2, \tau_h | ,
\]
where \( \phi_j \) satisfies
\[
\phi_{j,x} = -k_j \phi_j, \quad \phi_{j,xx} = k_j^2 \phi_j, \quad (k_j \in \mathbb{R}) , \tag{4.15a}
\]
\[
\phi_{j,t} = -\frac{1}{4k_j} \tilde{\phi}_j . \tag{4.15b}
\]
On the other hand, formula (4.11) satisfies condition (4.15a) but
\[ \phi_{j,t} = -\frac{1}{4k_j} \phi_j - \beta_j(t) \phi_j . \] (4.16)

So, noticing that the time evolution (4.16) is just a combination of (4.15b) and
\[ \phi_j = e^{\pi i/4} [e^{\theta_j + \pi i/4} - (-1)^j e^{-\theta_j - \pi i/4}] , \]
\[ \theta_j = k_j x + \int_0^t \beta_j(z) dz + \theta_j(0) , \quad (j = 1, 2, \ldots, N) \] (4.18)
solve the following bilinear equation:
\[ D_x D_t f \cdot f = -8i \sum_{j=1}^N \lambda_j \bar{g}_j^2 , \]
i.e.,
\[ f f_{xt} - f_x f_t = -4i \sum_{j=1}^N \lambda_j \bar{g}_j^2 . \] (4.19)

Due to the arbitrary term \( \int_0^t \beta_j(z) dz \) in (4.18), the derivatives \( f_t \) and \( f_{xt} \) are no longer to possess the compact and simple forms like before [17–20]. This arbitrary term, which results directly in the variation of the velocity of soliton, leads to the difficulties in the procedure of Wronskian verifications.

Now, expanding \( f = |N-1| \) and \( f_x = |N-2,N| \) by the \( j \)th row, we have
\[ f = e^{\pi i/4} \sum_{l=1}^N (-1)^j + l [e^{\theta_j + \pi i/4} - (-1)^j e^{-\theta_j - \pi i/4}]^{(l-1)} A_{j,l} , \quad (j = 1, 2, \ldots, N) \] (4.20)
\[ f_x = e^{\pi i/4} \left\{ \sum_{l=1}^{N-1} (-1)^j + l [e^{\theta_j + \pi i/4} - (-1)^j e^{-\theta_j - \pi i/4}]^{(l-1)} B_{j,l} \right. \\
+ (-1)^{j+N} [e^{\theta_j + \pi i/4} - (-1)^j e^{-\theta_j - \pi i/4}]^{(N)} B_{j,N} \right\} , \] (4.21)
where \( A_{j,l} \) and \( B_{j,l} \) are the minor determinants of \( f \) and \( f_x \), respectively. Obviously \( A_{j,N} = B_{j,N} \). It then follows from (4.20) and (4.21) that
\[ f_t = e^{\pi i/4} \sum_{j=1}^N \beta_j(t) \sum_{l=1}^N (-1)^j + l [e^{\theta_j + \pi i/4} + (-1)^j e^{-\theta_j - \pi i/4}]^{(l-1)} A_{j,l} , \] (4.22)
\[ f_{xt} = e^{\pi i/4} \sum_{j=1}^{N} \beta_j(t) \left\{ \sum_{l=1}^{N-1} (1)^{l+1} \left[ e^{\theta_j + \pi i/4} + (1)^{1} e^{-\theta_j - \pi i/4} \right] (l-1) B_{j,l} \right. \\
+ (1)^{l+N} \left[ e^{\theta_j + \pi i/4} + (1)^{1} e^{-\theta_j - \pi i/4} \right] (N) B_{j,N} \right\} \]  

(4.23)

Noticing that the fact
\[ e^{\theta_j + \pi i/4} - (1)^{1} e^{-\theta_j - \pi i/4} \]
\[ = (1)^{l-1} 2k_{j,l}^{-1} [(-1)^{h} - (-1)^{s}] \]

we have
\[ f_{xt} - f_{xt} = 2i \sum_{j=1}^{N} (1)^{l-1} \beta_j(t) \left\{ \sum_{l=1}^{N-1} \sum_{h=1}^{l-1} (1)^{l+h} k_{j,l}^{h-2} [(-1)^{h} - (1)^{N} A_{j,i} B_{j,h} \right. \\
- \sum_{l=1}^{N-1} (1)^{l+N} k_{j,l}^{l+N-2} [(-1)^{l} - (1)^{N} A_{j,l} B_{j,l} \right. \\
+ \sum_{l=1}^{N} (1)^{l+N} k_{j,l}^{l+N-2} [(-1)^{l} - (1)^{N} A_{j,N} B_{j,l} \right. \\
+ (1)^{N-1} 2k_{j}^{2N-1} A_{j,N}^2 \right\} \]

Further, replacing \( l + h \) by \( 2m + 1 \) leads to
\[ f_{xt} - f_{xt} = 4i \sum_{j=1}^{N} (1)^{l} \beta_j(t) (P_j + Q_j + R_j + S_j + T_j) , \]  

(4.24a)

where
\[ P_j = \sum_{m=1}^{[N-1/2]} k_{j}^{2m-1} \sum_{h=0}^{m-1} (A_{j,2h+1} B_{j,2(2m-h)} - A_{j,2(2m-h)} B_{j,2h+1}) , \]  

(4.24b)

\[ Q_j = \frac{1}{2} \left[ 1 + (-1)^{N} \right] k_{j}^{N-1} \]
\[ \times \left\{ \sum_{h=0}^{[N/2-2]} (A_{j,2h+3} B_{j,N-2h-2} - A_{j,N-2h-2} B_{j,2h+3}) - B_{j1} A_{j,N} \right\} , \]  

(4.24c)
\[ R_j = (-1)^N \sum_{m=\lceil N/2 \rceil + 1}^{N-2} k_j^{2m-1} \left[ \sum_{h=0}^{N-m-2} (A_{j,2(h+m)+3-N} B_{j,N-2h-2} - A_{j,N-2h-2}) \times B_{j,2(h+m)+3-N} + (A_{j,2m-N} - B_{j,2m-N+1}) A_{jN} \right], \]  
\[ (4.24d) \]

\[ S_j = (-1)^N k_j^{2N-3} (A_{j,N-2} - B_{j,N-1}) A_{jN} , \]  
\[ (4.24e) \]

\[ T_j = (-1)^N k_j^{2N-1} A_{jN}^2 . \]  
\[ (4.24f) \]

Now, for being simple and explicit, we only take \( N \) being even and \( j = 1 \) as an example. It is easy to find that

\[ A_{1,h} = |M(h,1,N)|, \quad (h = 1,2,\ldots,N) , \quad (4.25a) \]

\[ B_{1,h} = |M(h,1,N-1)|, \quad (h = 1,2,\ldots,N-1), \quad B_{1,N} = A_{1N} , \quad (4.25b) \]

where \( M(h,l) \) is defined by formula (4.5). It further appears that

\[ A_{1,h+1}B_{1,l+1} - A_{1,l+1}B_{1,h+1} = -|M(h,l)|A_{1,N}, \quad (0 \leq h < l \leq N-2) \]  
\[ (4.26) \]

with the help of the following familiar identity [17–20]:

\[ |M,a,b||M,c,d| - |M,a,c||M,b,d| + |M,a,d||M,b,c| = 0 , \]

where \( M \) is an \((N-1) \times (N-3)\) matrix and \( a, b, c \) and \( d \) represent \( N-1 \) column vectors.

For \( P_1 \), in the case of \( 2h < 2(m-h) - 1 \), i.e., \( 0 \leq 2h \leq m-1 \), by means of the identity (4.26), we have

\[ A_{1,2h+1}B_{1,2(m-h)} - A_{1,2(m-h)}B_{1,2h+1} = (1)^s|M(s,2m-s-1)|A_{1N} , \]  
\[ (4.27) \]

where \( s = 2h \) is an even number and \( 0 \leq s \leq m-1 \). On the other hand, in the case of \( 2h > 2(m-h) - 1 \), i.e., \( m \leq 2h \leq 2m-2 \), we also have (4.27) but \( s = 2(m-h) - 1 \) is odd and \( 1 \leq s \leq m-1 \). That means

\[ P_1 = - \sum_{m=1}^{N/2-1} k_j^{2m-1} \sum_{s=0}^{m-1} (-1)^s|M(s,2m-s-1)|A_{1N} . \]  
\[ (4.28) \]

For \( Q_1 \), similar to the treatment of \( P_1 \) and noticing that \( B_{11} = -|M(0,N-1)| \), we can find that

\[ Q_1 = -k_j^{2m-1} \sum_{s=0}^{m-1} (-1)^s|M(s,2m-s-1)|A_{1N}, \quad \left( m = \frac{N}{2} \right) . \]  
\[ (4.29) \]
Next, similarly,

\[ R_1 = -(-1)^N \sum_{m=N/2+1}^{N-2} k_1^{2m-1} \sum_{s=-1}^{N-m-1} (-1)^s |M(2m - N + s, N - s - 1)| A_{1N}. \]

(4.30)

So, based on the result of Proposition 2, the coefficient of $\beta_1(t)$ in (4.24a) is just

\[
4i k_1 \left[ \sum_{m=1}^{N/2} k_1^{2m-2} \sum_{s=0}^{m-1} (-1)^s |M(s, 2m - s - 1)| A_{1N} 
+ (-1)^N \sum_{m=N/2+1}^{N} k_1^{2m-2} \sum_{s=-1}^{N-m-1} (-1)^s |M(2m - N + s, N - s - 1)| A_{1N} \right] 
= 4i k_1 \left[ \sum_{m=1}^{N} (-1)^{m-1} k_1^{2m-2} \left( \sum_{2 \leq j_1 < j_2 < \cdots < j_{N-m} \leq N} k_{j_1}^2 k_{j_2}^2 \cdots k_{j_{N-m}}^2 \right) \right] A_{1,N}^2 
= 4i k_1 \left[ \prod_{j=2}^{N} (k_j^2 - k_1^2) \right] A_{1,N}^2. 
\]

(4.31)

Thus we have completed the proof of the Proposition 3. \(\square\)

Conclusions

We have present a general way to derive the hierarchy of the SGESCS. We also give the bilinear form of the SGESCS, from which the $N$-soliton solutions are obtained through Hirota method and Wronskian technique, respectively. Some novel determinantal identities are developed to treat the nonlinear term in the time evolution and finish the Wronskian verifications.

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Appendix A. The proof of Eq. (3.7)

We will show (3.7) and (4.2) are the same for recovering the $N$-soliton solutions from transformation (3.1).
By virtue of the addition rule of determinants, \( f = |N|^{-1} \) with (4.11) can be represented by the sum of \( 2^N \) Vandermonde determinants. So we have

\[
\begin{align*}
\begin{bmatrix}
N \prod_{j=1}^{N} e^{\varepsilon_j (\zeta_j + (\pi/4)i)} & \begin{bmatrix} [N/2] \end{bmatrix} e_{2h} \end{bmatrix} \Delta(\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_N k_N) \\
(\varepsilon_1 k_1 - \varepsilon_j k_j)
\end{bmatrix}
\end{align*}
\]

\[
= (e^{(\pi/4)i})^N \sum_{\varepsilon = \pm 1} \prod_{j=1}^{N} e^{\varepsilon_j (\zeta_j + (\pi/4)i)} \begin{bmatrix} [N/2] \end{bmatrix} e_{2h} \prod_{1 \leq j < l \leq N} (\varepsilon_1 k_1 - \varepsilon_j k_j)
\]

\[
= (e^{(\pi/4)i})^N \sum_{\varepsilon = \pm 1} \prod_{j=1}^{N} e^{\varepsilon_j (\zeta_j + (\pi/4)i)} \prod_{1 \leq j < l \leq N} (k_1 - \varepsilon_j k_j)
\]

where \( \Delta(\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_N k_N) \) denotes an \( N \times N \) Vandermonde determinant with the entries \( \varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_N k_N \), and the sum over \( \varepsilon = \pm 1 \) refers to each of the \( \varepsilon_j = 1, -1 \), \( j = 1, 2, \ldots, N \). Then noticing that

\[
k_l - \varepsilon_i \varepsilon_j k_j = (k_1 - k_j) \left( \frac{k_l - k_j}{k_l + k_j} \right)^{1/2(\varepsilon_i \varepsilon_j - 1)}
\]

and setting \( \mu_j = (1 + \varepsilon_j)/2 \) lead to

\[
f = \left( \prod_{j=1}^{N} e^{-\zeta_j} \right) \left( \prod_{1 \leq j < l \leq N} (k_1 - k_j) \right)
\]

\[
\times \sum_{\mu = 0, 1} \left\{ \prod_{1 \leq j < l \leq N} \left( \frac{k_l - k_j}{k_l + k_j} \right)^{-\mu_j - \mu_l + 2\mu_j \mu_l} \right\} \prod_{j=1}^{N} e^{2\mu_j (\zeta_j + (\pi/4)i)}
\]

(A.1)

Since

\[
\prod_{1 \leq j < l \leq N} \left( \frac{k_l - k_j}{k_l + k_j} \right)^{-\mu_j - \mu_l} = \exp \left\{ -\sum_{j=1}^{N} \left( \mu_j \sum_{l=1, l \neq j}^{N} \frac{A_{jl}}{2} \right) \right\},
\]

we can arrive at

\[
f = \left( \prod_{j=1}^{N} e^{-\zeta_j} \right) \left( \prod_{1 \leq j < l \leq N} (k_1 - k_j) \right)
\]

\[
\times \sum_{\mu = 0, 1} \exp \left\{ \sum_{j=1}^{N} \mu_j \left( 2\eta_j + \frac{\pi}{2} i \right) + \sum_{1 \leq j < l \leq N} \mu_j \mu_l A_{jl} \right\},
\]

(A.3)

where

\[
\eta_j = \zeta_j - \frac{1}{4} \sum_{l=1, l \neq j}^{N} A_{jl}.
\]

(A.4)
We can deal with $g_h$ in a similar way to $f$, but we should first rewrite $\phi_j$ ($j > h$) as

$$
\phi_j = -ie^{\frac{\pi}{4} i} [e^{(\xi_j + (\pi/2) i) + (\pi/4) i} + (-1)^j e^{-(\xi_j + (\pi/2) i) - (\pi/4) i}] = \psi_j, \quad (j > h)
$$

and let $\psi_j$ denote the above new form of $\phi_j$ ($j > h$). In this case, it is not difficult to derive

$$
g_h = \sqrt{\beta_h(t)} \prod_{j=1}^{h-1} (k^2_j - k^2_h) \cdot \prod_{j=h+1}^{N} (k^2_j - k^2_h) \cdot \sum_{\mu=0,1} \exp \left\{ \mu_h \eta_h + \sum_{j=1}^{h-1} \mu_j \left( 2\eta_j + \frac{1}{2} A_{jh} + \frac{\pi}{2} i \right) \right. \\
+ \sum_{j=h+1}^{N} \mu_j \left( 2\eta_j + \frac{1}{2} A_{jh} - \frac{\pi}{2} i \right) + \sum_{1 \leq j < l \leq N, j, l \neq h} \mu_j \mu_l A_{jl} \right\},
$$

which means (3.7) and (4.12) are the same for recovering the $N$-soliton solutions from transformation (3.1) and the conjectured formula (3.7) is right.

References