The N-soliton solutions for the non-isospectral mKdV equation

Yi Zhang\textsuperscript{a,b,*}, Shu-fang Deng\textsuperscript{b}, Da-jun Zhang\textsuperscript{b}, Deng-yuan Chen\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Zhejiang Normal University, Jinhua 321004, People’s Republic of China
\textsuperscript{b}Department of Mathematics, Shanghai University, Shanghai 200436, People’s Republic of China

Received 15 December 2003; received in revised form 23 February 2004

Abstract

The bilinear form of the non-isospectral mKdV equation is given and the exact \(N\)-soliton solutions are obtained through Hirota method and Wronskian technique, respectively. © 2004 Elsevier B.V. All rights reserved.

\textbf{PACS:} 02.30.Ik; 05.45.Yv

\textbf{Keywords:} The non-isospectral mKdV equation; Hirota method; Wronskian technique

1. Introduction

It is well known that Hirota bilinear method has provided us with a powerful tool to find exact solutions for a variety of non-linear evolution equations [1]. Once a non-linear equation is written in bilinear forms by a dependent variable transformation, then multi-soliton solutions are obtained in the form of an \(N\)th-order polynomial in \(N\) exponentials.

On the other hand, there exists another method called Wronskian technique for which the \(N\)-soliton solutions consisting of quadratic expression in terms of determinants are satisfied. Satsuma [2], Freeman and Nimmo [3,4] have shown that identity may be verified directly without recourse to mathematical induction and applied it to a variety of soliton equations. The key of this technique is an extension use of \(N\)-soliton solutions expressed in terms of the Wronskian determinants and Hirota bilinear formula is reduced to the Laplace expression of a certain determinant which is equal to zero.

\* Corresponding author. Department of Mathematics, Zhejiang Normal University, Jinhua 321004, People’s Republic of China. Fax: +86-21-66133292.
\textit{E-mail address:} zy2836@163.com (Y. Zhang).

0378-4371/ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2004.03.008
The purpose of the present paper is to derive the $N$-soliton solutions for the non-isospectral modified KdV(mKdV) equation by applying Hirota method and Wronskian technique. It is noted that the two direct methods depend on the bilinear forms of the non-isospectral mKdV equation. We also point out that the solutions of the bilinear equations obtained through the above methods are differential, which is to be seen by comparing. In the case of isospectral mKdV equation, the bilinear form was obtained by Hirota and Satsuma [5], and the $N$-soliton solution in Wronskian form was constructed by Nimmo and Freeman [6]. Applying different methods, Lin et al. [7] and Zhang [8] have studied the mKdV equation with self-consistent sources and obtained the explicit soliton solutions, respectively. To our knowledge, it is the first attempt to seek soliton solutions by Hirota and Wronskian technique for non-isospectral mKdV equation, with loss and non-uniformity terms [9].

This paper is organized as follows: In Section 1, we deduce the hierarchy of the non-isospectral mKdV equation. In Section 2, we give the bilinear form of the non-isospectral mKdV equation and solve it by means of Hirota method. In Section 3, Wronskian solution is constructed and verified. Section 4 is devoted to concluding remarks.

2. Non-isospectral mKdV hierarchy

In this section, we briefly present a description of the hierarchy for the non-isospectral mKdV equation. Consider the following eigenvalue problem [10]:

\[ \phi_{xx} + 2v \phi_x = \lambda \phi , \]  
\[(2.1a)\]

where $v = v(x,t)$ is a potential function and $\lambda$ is the spectral parameter and its $t$-evolution part

\[ \phi_t = A \phi + B \phi_x , \]  
\[(2.1b)\]

where $A, B$ are scalar functions independent of $\phi$.

The compatibility condition of (2.1a) and (2.1b) gives rise to a zero curvature equation

\[ A_{xx} + 2v A_x + 2\lambda B_x - \lambda_t = 0 , \]  
\[(2.2a)\]

\[ v_t + \frac{1}{2} B_{xx} - v B_x - v_x B + A_x = 0 . \]  
\[(2.2b)\]

Eliminating $B$, we have

\[ \begin{split} 
\lambda v_t &= \left( \frac{1}{4} \dot{v}^2 - v^2 - v_x \dot{v}^{-1} v \right) A_x + \frac{1}{2} (xv)_x \lambda_t - \lambda A_x + \lambda v_x B_0 , 
\end{split} \]  
\[(2.3)\]

where $B_0$ is a constant, $\dot{\phi} = \frac{\partial}{\partial t}$, $\dot{v}^{-1} \dot{\phi} = \ddot{\phi} \dot{v}^{-1} = 1$. 


The substitution of
\[ A = \sum_{j=1}^{n+1} a_j \lambda^{n+1-j} \] (2.4)
into Eq. (2.3) leads to the following recursion relations:
\[ v_t = \left( \frac{1}{4} \partial^2 - v^2 - v_x \partial^{-1} v \right) a_{n,x}, \] (2.5a)
\[ a_{j+1,x} = \left( \frac{1}{4} \partial^2 - v^2 - v_x \partial^{-1} v \right) a_{j,x} \quad j = 1, 2, \ldots, n-1. \] (2.5b)

Under the condition of the non-isospectral, one may take \( \lambda_t = \frac{1}{2} (4\lambda)^{n+1} \) and \( B_0 = 0 \). Then we obtain
\[ a_{1,x} = 4^n (xv)_x. \] (2.5c)

From Eqs. (2.5), one gets
\[ a_{j+1,x} = 4^{n-j} F_j (xv)_x, \quad (j = 1, 2, \ldots, n-1). \] (2.6)

So we derive the hierarchy of the non-isospectral mKdV equation
\[ v_t = F^n (xv)_x \quad (n = 1, 2) \] (2.7)

As a special case, Eq. (2.7) for \( n = 1 \) gives rise to the following:
\[ v_t - x(v_{xxx} - 6v_x v^2) - 3v_{xx} + 4v^3 + 2v_x \partial^{-1} v^2 = 0. \] (2.8)

By using the transformation \( t \rightarrow -t, v \rightarrow iv \), into (2.8), and deduce that Eq. (2.8) can be transformed into
\[ v_t + x(v_{xxx} + 6v_x v^2) + 3v_{xx} + 4v^3 + 2v_x \partial^{-1} v^2 = 0. \] (2.9)

3. Bilinear form and Hirota method

In this part, we present the bilinear form of Eq. (2.9) and derive the soliton solutions by Hirota method. By the dependent variable transformation [3,4]
\[ v = i \left( \ln \frac{\tilde{f}}{f} \right)_x, \] (3.1)
where \( \tilde{f} \) is the complex conjugates of \( f \), Eq. (2.9) can be transformed into the bilinear form
\[ (D_x + xD_x^3) \tilde{f} \cdot f + 2(\tilde{f} \cdot f_{xx} - f_{xx} \tilde{f}) = 0, \] (3.2a)
\[ D_x^2 \tilde{f} \cdot f = 0, \] (3.2b)
where $D$ is the well-known Hirota bilinear operator defined by

$$\begin{align*}
D_n^m a \cdot b &= (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x,t)b(x',t')|_{x'=x,t'=t}.
\end{align*}$$

Expanding $f$, $\tilde{f}$ as follows

$$\begin{align*}
f &= 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots, \\
\tilde{f} &= 1 + \varepsilon \tilde{f}^{(1)} + \varepsilon^2 \tilde{f}^{(2)} + \cdots.
\end{align*}$$

The bilinear equations (3.3) further imply that

$$\begin{align*}
\tilde{f}^{(1)}_t + x \tilde{f}^{(1)}_{xxx} - \tilde{f}^{(1)}_t - x f^{(1)}_{xxx} + 2 \tilde{f}^{(1)}_x - 2 f^{(1)}_x &= 0, \\
\tilde{f}^{(2)}_t + x \tilde{f}^{(2)}_{xxx} - f^{(2)}_t - x f^{(2)}_{xxx} + 2 \tilde{f}^{(2)}_x - 2 f^{(2)}_x &= -(D_t + x D_x^3) f^{(1)} \cdot f^{(1)} - 2(f^{(1)} f^{(1)} f^{(1)} - f^{(1)} f^{(1)} f^{(1)}), \\
\tilde{f}^{(1)}_{xx} + f^{(1)}_x &= 0, \\
\tilde{f}^{(2)}_{xx} + f^{(2)}_x &= -D_x^2 \tilde{f}^{(1)} \cdot f^{(1)},
\end{align*}$$

Substituting Eqs. (3.3) into (3.4) and equate coefficient of $\varepsilon$. Starting with $f^{(1)} = \omega_1(t)e^{\xi_1 + \pi/2}i$, $\xi_1 = k_1(t)x + \xi_1^{(0)}$, and $\xi_1^{(0)}$ is an arbitrary constant, where $\omega_1(t), k_1(t)$ is real function with respect to $t$ and $\omega_{1,i}(t) = -2\omega_1(t)k_1^2(t)$, $k_{1,i}(t) = -k_1^2(t)$. By utilizing the Hirota method, we have

$$f^{(1)} = \omega_1(t)e^{\xi_1 + \pi/2}i$$

and

$$f^{(j)} = 0, \quad j = 2, 3, \cdots.$$ 

Inserting (3.5) into (3.1), we have an explicit form of the one-soliton solution

$$v = \frac{2k_1(t)\omega_1(t)e^{\xi_1}}{1 + \omega_1^2(t)e^{2\xi_1}}.$$ 

In a manner similar, if we take

$$f^{(1)} = \omega_1(t)e^{\xi_1 + \pi/2}i + \omega_2(t)e^{\xi_2 + \pi/2}i, \quad \xi_j = k_j(t)x + \xi_j^{(0)}, \quad (j = 1, 2)$$

then

$$f^{(2)} = \omega_1(t)\omega_2(t)e^{\xi_1 + \xi_2 + \pi i + A_{12}},$$

$$f^{(j)} = 0, \quad j = 3, 4, \cdots$$
and
\[ e^{A_{12}} = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2. \] (3.7d)

Therefore the two-soliton solution is obtained from Eq. (3.1) here
\[ f = 1 + \omega_1(t)e^{\xi_1 + \pi i} + \omega_2(t)e^{\xi_2 + \pi i} + \omega_1(t)\omega_2(t)e^{\xi_1 + \xi_2 + \pi i + A_{12}}, \] (3.8a)
\[ \tilde{f} = 1 - \omega_1(t)e^{\xi_1 + \pi i} - \omega_2(t)e^{\xi_2 + \pi i} \]
\[ + \omega_1(t)\omega_2(t)e^{\xi_1 + \xi_2 + \pi i + A_{12}}. \] (3.8b)

This process can be continued to the three-, four-soliton solutions and so on. Generally, the \( N \)-soliton solutions can be denoted by
\[ f = \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j + \ln(\omega_j(t)) + \frac{\pi i}{2} \right) + \sum_{1 \leq j < l} \mu_j \mu_l A_{jl} \right], \] (3.9a)
with
\[ \xi_j = k_j(t)x + \xi_j(0), \] (3.9b)
\[ \omega_j(t), k_j(t) \] satisfy the following equation:
\[ \omega_{j,t}(t) = -2\omega_j(t)k_j^2(t), \] (3.9c)
\[ k_{j,t}(t) = -k_j^3(t), \] (3.9d)
and
\[ e^{A_{jj}} = \left( \frac{k_j - k_l}{k_j + k_l} \right)^2, \] (3.9e)
where the sum is taken over all possible combinations of \( \mu_j = 0, 1 \) \( (j = 1, 2, \ldots, N) \).

4. Wronskian solution

In this section, we show that the solutions for the non-isospectral mKdV equation are given in terms of the Wronskian. In the following, we use the abbreviated notion of Freeman and Nimmo [3,4] for the Wronskian and its derivatives. The Wronskian solution for Eqs. (3.2) is given by
\[ f = W(\phi_1, \phi_2, \ldots, \phi_N) \]
\[ = \begin{vmatrix} \phi_1 & \hat{\phi}_1 & \ldots & \hat{\phi}^{N-1} \phi_1 \\ \phi_2 & \hat{\phi}_2 & \ldots & \hat{\phi}^{N-1} \phi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N & \hat{\phi}_N & \ldots & \hat{\phi}^{N-1} \phi_N \end{vmatrix} = |0, 1, \ldots, N - 1| = |N - 1|, \] (4.1)
where the entries $\phi_j(j = 1, 2, \ldots, N)$ satisfies
\begin{equation}
\phi_{j,x} = \varkappa_j(t) \tilde{\phi}_j ,
\tag{4.2a}
\end{equation}
\begin{equation}
\phi_{j,t} = -4x \phi_{j,xxx} + (4N - 6) \phi_{j,xx} ,
\tag{4.2b}
\end{equation}
where $\varkappa_j(t)$ satisfies
\begin{equation}
k_{j,t}(t) = -8\varkappa_j^2(t) .
\tag{4.2c}
\end{equation}
From (4.2a), it is readily concluded that
\begin{equation}
\phi_{j,xx} = \varkappa_j^2(t) \phi_j
\tag{4.3a}
\end{equation}
and
\begin{equation}
\bar{\phi}_j = \varkappa_j(t) \tilde{\varkappa}^{-1} \phi_j .
\tag{4.3b}
\end{equation}
To finish the verification we should calculate various differentiation which appear in Eqs. (3.2). First, we list the derivative of $f$ and $\tilde{f}$, where the procedure is in accordance with in Ref. [11].
\begin{equation}
f_x = |N - 2, N| , \quad f_{xx} = (|N - 3, N - 1, N| + |N - 2, N + 1|) ,
\end{equation}
\begin{equation}
f_{xxx} = (|N - 4, N - 2, N - 1, N| + 2|N - 3, N - 1, N + 1| + |N - 2, N + 2|) ,
\end{equation}
\begin{equation}
\tilde{f}_x = A| - 1, N - 3, N - 1| ,
\end{equation}
\begin{equation}
\tilde{f}_{xx} = A(| - 1, N - 4, N - 2, N - 1| + | - 1, N - 3, N|) ,
\end{equation}
\begin{equation}
\tilde{f}_{xxx} = A(| - 1, N - 5, N - 3, N - 2, N - 1| + 2 |
-1, N - 4, N - 2, N| + | - 1, N - 3, N + 1|) ,
\end{equation}
where $A = \prod_{j=1}^{N} \varkappa_j(t) .
\end{equation}

In the case of non-isospectral, the derivative $f_t$ and $\tilde{f}_t$ are no longer possess the simple forms as usual [6,9]. The $l$th-order derivatives of $\phi_{j,t}$ may easily be solved to Eq. (4.2b), we have
\begin{equation}
\phi_{j,t}^{(l)} = -4x \phi_{j,t}^{(l+3)} + (4N - 6 - 4l) \phi_{j,t}^{(l+2)} .
\tag{4.4}
\end{equation}
It then follows that
\begin{equation}
f_t = -4x (|N - 4, N - 2, N - 1, N| - |N - 3, N - 1, N + 1| + |N - 2, N + 2|)
-2|N - 3, N - 1, N| - 2|N - 2, N + 1| .
\tag{4.5}
\end{equation}
Next we calculate \( \tilde{f}_t \). Eq. (4.2b) can be transformed

\[
\tilde{\phi}_{j, t} = -8x_j(t) \tilde{\phi}_{j, t} + \phi_j(t) \tilde{\phi}_{j, t}^{-1},
\]

then the derivatives of \( l \)-th order of \( \tilde{\phi}_j(t) \) as follows:

\[
\tilde{\phi}_j(t)^{(l)} = -8x_j(t) \tilde{\phi}_j(t)^{(l)} + \phi_j(t) [ -4x \tilde{\phi}_j(t)^{(l+3)} ] + (4N - 4l - 2) \tilde{\phi}_j(t)^{(l+2)}.
\]

So we have

\[
\tilde{f}_t = -4Ax(-1, N - 5, N - 3, N - 2, N - 1 | -1, N - 5, N - 2, N | + |N - 3, N + 1| -2 | -1, N - 4, N - 2, N - 1 | - 2 | -1, N - 3, N |.
\]

Noting that the facts

\[
\sum_{j=1}^{N} \alpha_j^2(t) | - 1, N - 2 | = - |N - 4, N - 2, N - 1 | + | -1, N - 3, N |,
\]

\[
\sum_{j=1}^{N} \alpha_j^2(t) | - 1, N - 3, N - 1 | = - | -1, N - 5, N - 3, N - 2,
\]

\[
N - 1 | + | -1, N - 3, N + 1 |,
\]

\[
\sum_{j=1}^{N} \alpha_j^2(t) |N - 2, N | = - |N - 4, N - 2, N - 1, N | + |N - 2, N + 2 |,
\]

\[
\sum_{j=1}^{N} \alpha_j^2(t) |N - 1 | = - |N - 3, N - 1, N | + |N - 2, N + 1 |.
\]

Substitution of the variety derivatives of \( f \) and \( \tilde{f} \) on the left-hand side of Eq. (3.2a), we have

\[
-6A(-1)^{N - 2}x(|N - 2, N | - 1, N - 4, N - 2, N - 1 | + | -1, N - 2 | |N - 4, N - 2, N - 1, N | - | -1, N - 4, N - 2, N | |N - 1 | + |N - 3, N - 1 | + |N - 2 | |1, N - 2 | - | -1, N - 3, N - 1 | |N - 3, N - 1, N |.
\]
\[ -|N-1| - 1, N-5, N-3, N-2, N-1| \]
\[ = -3A(-1)^{N-2} \]
\[ \times \begin{vmatrix} N-4 & N-2 & 0 & 0 & -1 & N-3 & N-1 & N \\ 0 & 0 & N-4 & N-2 & -1 & N-3 & N-1 & N \\ N-3 & 0 & -1 & N-2 & N-1 & N+1 \end{vmatrix} \]
\[ + \begin{vmatrix} N-3 & 0 & -1 & N-2 & N-1 & N+1 \\
0 & N-3 & -1 & N-2 & N-1 & N \end{vmatrix}, \tag{4.13} \]
which is easily seen to be zero.

In a very similar way, Eq. (3.2b) may be shown to give
\[ 2A(|N-2, N+1| - 1, N-2| + | - 1, N-4, N-2, N-1|N-1| \]
\[ - | - 1, N-3, N-1|N-2, N| \), \tag{4.14} \]
which is the expansion by \( N \times N \) minors of the \( (2N-1) \times (2N-1) \) determinant
\[ (-1)^{N-2}A \begin{vmatrix} N-3 & 0 & -1 & N-2 & N-1 & N \\ 0 & N-3 & -1 & N-2 & N-1 & N \end{vmatrix}, \tag{4.15} \]
which may be shown to be identically zero. Hence we have proved that the Wronskian solution in Eq. (3.2) gives the solution for non-isospectral mKdV equation.

5. Concluding remarks

The hierarchy of the non-isospectral mKdV equation is deduced. We have obtained the exact \( N \)-soliton solutions of the non-isospectral mKdV equation through the Hirota method and Wronskian technique, respectively. In the case of the non-isospectral, however, the consistence of the \( N \)-soliton solutions above will no longer hold. This point is in striking contract to the isospectral case. The essence of the methods lies in the bilinear algebraic structure. On the basis of bilinear formalism, we can construct a Bäcklund transformation for the non-isospectral mKdV equation, which will be reported in the subsequent study.

Acknowledgements

This project is supported by the National Science Foundation of China and the Special Funds for Major Specialities of Shanghai Education Committee.

References