Exact Solutions for the Nonisospectral Kadomtsev–Petviashvili Equation

Shu-fang DENG*, Da-jun ZHANG1† and Deng-yuan CHEN1

Institute of Mathematics, Fudan University, Shanghai 200433, People’s Republic of China
1Department of Mathematics, Shanghai University, Shanghai 200446, People’s Republic of China

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The nonisospectral Kadomtsev–Petviashvili (KP) equation is solved by the Hirota method and Wronskian technique. Exact solutions that possess soliton characters with nonisospectral properties are obtained. In addition, rational and mixed solutions are derived. We also obtain a new molecular equation that admits a solution in the Wronskian form.

KEYWORDS: nonisospectral KP equation, Hirota method, Wronskian technique

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In 1971, Hirota1) first proposed the formal perturbation technique, called the Hirota method, to obtain N-soliton solutions of the KdV equation. The solution obtained in this manner is written as a polynomial of exponential functions. The soliton solution can also be written in the Wronskian form, which was first introduced by Satsuma2) in 1979. Further more, Freeman and Nimmo3) developed a general verification procedure, which we call the Wronskian technique. Many soliton equations have been revealed to be exactly solvable by these two direct methods.4–7)

The Hirota method can also be applied to equations with nonisospectral properties, for example, the KdV equation with loss and nonuniformity terms.8)

In this letter, we would like to consider the nonisospectral Kadomtsev–Petviashvili (KP) equation9) in the above two direct methods. The bilinear form of the nonisospectral KP equation is given, and one- and two-soliton solutions are obtained by the standard Hirota method. A general formula that denotes higher order solutions is also given. We provide some figures to show the shapes and motions of some solutions we have obtained. We can observe in these figures the soliton characters with nonisospectral properties. We also derive new solutions in the Wronskian form. Further more we obtain rational and mixed rational-soliton solutions. It is interesting that, as a by-product, a new molecular equation that admits a solution in the Wronskian form is found.

Here, we present the bilinear form of the nonisospectral KP equation and derive the soliton solutions by the Hirota method.

The nonisospectral KP equation9) is

\[ 4u_t + 3\theta u_{xxx} + 6u_xu_t + 2ux_x u_t + 4x^{-1}u_t = 0, \]

and its Lax pair is

\[ \phi_2 = \phi_0 + 2u\phi, \]

\[ \phi_1 = \left[ \phi_2 + 3u\phi_0 + \frac{3}{2}(\partial_t^{-1}u_x + u_x)\phi \right] + \frac{1}{2}u(\partial_t^{1}u_x + 2u\phi) + \frac{1}{2}\phi_0 + \frac{1}{2}(\partial_t^{-1}u)\phi. \]

With the help of the dependent variable transformation

\[ u = 2(\ln f)_{xx}, \]

eq (1) can be transformed into the bilinear form

\[ 4D_xD_yf \cdot f + y(D_x^2f \cdot f + 3D_y^2f \cdot f) + 2xD_xD_yf \cdot f + 4f_xf_y = 0, \]

where \( D \) is the well-known Hirota bilinear operator

\[ D_x^kD_y^l a \cdot b = (\partial_x - \partial_y)^k(\partial_x - \partial_y)^l(a - b)^{p_0}(\partial_x - \partial_y)^p. \]

This bilinear equation further suggests

\[ 4f^{(1)}_t + y(f^{(1)}_{xxx} + 3f^{(1)}_{yy}) + 2xf^{(1)}_x + 2y^{(1)} = 0, \]

\[ 8f^{(2)}_t + y(2f^{(2)}_{xxx} + 6f^{(2)}_{yy}) + 4xf^{(2)}_x + 4y^{(2)} = -4D_xD_yf^{(1)} \cdot f^{(1)} - y(D_x^2f^{(1)} \cdot f^{(1)} + 3D_y^2f^{(1)} \cdot f^{(1)}) - 2xD_xD_yf^{(1)} \cdot f^{(1)} - 4f^{(1)}_x f^{(1)} + \cdots, \]

under the perturbation expansion

\[ f(x, y, t) = 1 + f^{(1)}e^t + f^{(2)}e^2 + f^{(3)}e^3 + \cdots. \]

Taking

\[ f^{(1)} = e^{\theta_1}, \quad \theta_1 = K_1(t)[x + P_1(t)y] + \theta_1^{(0)}, \]

from eq. (5), we obtain

\[ K_{1,t}(t) = -\frac{1}{2}K_1(t)P_1(t), \]

\[ P_{1,t}(t) = -\frac{1}{4}K_1^2(t) - \frac{1}{4}P_1^2(t), \]

and

\[ f^{(j)} = 0, \quad j = 2, 3, \ldots. \]

Further more, eq. (7b) give the solutions

\[ K_1(t) = \frac{-8c_1}{4c_1^2 - t^2}, \quad P_1(t) = \frac{-4t}{4c_1^2 - t^2}. \]

Thus, the one-soliton solution for the nonisospectral KP equation is

\[ u = \frac{K_1^2(t)}{2} \text{sech} \frac{\theta_1}{2}. \]

Its shape and motion is shown in Fig. 1. For any fixed time \( t \), the wave shows the characteristic of \( u \to 0 \) as \( x, y \to \pm \infty \). The amplitude of the wave, i.e., \( K_1^2(t)/2 \), tends to be zero as \( t \to \pm \infty \). This nonisospectral characteristic is caused directly by the dominant term \( K_1(t) = 8c_1/(4c_1^2 - t^2) \) in
Therefore, the two-soliton solution is obtained from eq. (3),

\[
\theta_j = K_j(t)[x + P_j(t)y] + \theta_j^{(0)}, \quad (j = 1, 2)
\]

where

\[
f^{(2)} = e^{\theta_1 + \theta_2 + A_2},
\]

and

\[
f^{(j)} = 0, \quad j = 3, 4, \ldots.
\]

Therefore, the two-soliton solution is obtained from eq. (3), where

\[
f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_1 + \theta_2 + A_2}.
\]

This process can be extended to the three-soliton solution, four-soliton solution and so on. Generally, we obtain

\[
f = \sum_{\epsilon = 0, 1} \exp \left[ \sum_{j=1}^{N} \epsilon_j \theta_j + \sum_{j<l} \epsilon_j \epsilon_l \phi_{jl} \right],
\]

\[
\theta_j = K_j(t)[x + P_j(t)y] + \theta_j^{(0)},
\]

\[
K_j(t) = -\frac{1}{2} K_j(t) P_j(t), \quad P_j(t) = -\frac{1}{4} K_j^2(t) - \frac{1}{4} P_j^2(t),
\]

\[
K_j(t) = \frac{8c_j}{4c_j^2 - t^2}, \quad P_j(t) = \frac{-4t}{4c_j^2 - t^2},
\]

\[
e^{A_j} = \frac{[K_j(t) - K_j(t)]^2 - [P_j(t) - P_j(t)]^2}{[K_j(t) + K_j(t)]^2 - [P_j(t) - P_j(t)]^2},
\]

where the sum is obtained over all possible combinations of \( \epsilon_j = 0, 1 \) (\( j = 1, 2, \ldots, N \)).
Thus it is easy to prove that eq. (12) with eq. (13) can solve eq. (4) in a manner similar to that in ref. 3.

The explicit solution can be obtained by setting

\[ \phi_j = \alpha_j^+ A_j(t)e^{\xi_j} + \alpha_j^- B_j(t)e^{-\eta_j}, \quad (j = 1, 2, \ldots, N) \]  

(15a)

where \( \xi_j \) and \( \eta_j \) are described as

\[ \xi_j = k_j(t) \alpha_j + k_j^2(t) \beta_j + e^{\xi_j(0)}, \]
\[ \eta_j = q_j(t) \alpha_j + q_j^2(t) \beta_j + e^{\eta_j(0)}, \]

(15b)

\[ k_j(t) = -\frac{1}{2} k_j^2(t), \quad \eta_j = \frac{1}{2} q_j^2(t), \]
\[ k_j(t) = \frac{2}{c_j + t}, \quad \eta_j = \frac{2}{c_j - t}. \]

(15c)

and

\[ A_j(t) = (2c_j + t)^{-1}, \quad B_j(t) = (2c_j - t)^{-1}. \]

(15d)

If we take \( \alpha_j^+ = 1 \) and \( \alpha_j^- = -(1)^{j-1} \), similar to ref. 10, the Wronskian (12) can be written as

\[
\begin{align*}
& f = \frac{1}{N} \left[ \sum_{j=1}^{N} (q_j(t) - q_j^2(t)) \exp \left[ \sum_{j=1}^{N} (\eta_j - \ln B_j(t)) \right] \right] \\
& \times \sum_{e=0,1} \exp \left[ \sum_{j=1}^{N} e \xi_j + \sum_{1 \leq j < l \leq N} e \xi_j A_j^l \right], \\
& \theta_j' = \xi_j + \eta_j, \quad \xi_j = \xi_j + \ln A_j(t) + \sum_{j \neq l} [k_j(t) + q_j(t)], \\
& \eta_j' = \eta_j + \ln B_j(t) \\
& + \sum_{j \neq l} [q_j(t) - q_j^2(t)]^{-1} + \sum_{j \neq l} [q_j(t) - q_j^2(t)]^{-1}, \\
& e^{\phi_j} = \frac{[k_j(t) - k_j^2(t)][q_j(t) - q_j^2(t)]}{[k_j(t) + q_j(t)][k_j^2(t) + q_j(t)]}, \\
& e^{\phi_j} = e^{\phi_j}. \\
\end{align*}
\]

(16a)

(16b)

(16c)

and \( e^{\phi_j} \) is just \( e^{\phi_j} \) given as eq. (11e). Thus, we give another form of solution (12). However, the solutions (16) and (11) are slightly different in the covering of the N-soliton solution from the transformation (3). One can find that there is a time-dependent initial phase in each \( \theta_j' \) in the solution (16), which is different from eq. (11).

We can also generate other solutions in the Wronskian form. If we take

\[ Q_{j,l} = \frac{\partial}{\partial c_j^l} \phi_j, \quad j = 1, 2, \ldots, N, \quad l = 0, 1, 2, \ldots, \]

(17)

where \( \phi_j \) is given as eq. (15a). It is easy to see that eq. (17) is a solution of eqs. (13a) and (13b). Thus, the Wronskian

\[ f = W(Q_{j,1}, Q_{j,2}, \ldots, Q_{j,N}) \]

(18)

solves the the bilinear equation (4).

In general, eqs. (18) and (19) denote mixed solutions. If we take \( \alpha_j^+ = 1 \) and \( \alpha_j^- = 0 \) in (15a), the Wronskian (18) can denote the rational solutions of the nonisospectral KP equation. Here are some simple examples:

\[ f = W(Q_{i,1}, Q_{i,2}) \]
\[ = 2[-k_i(t) + x k_i^2(t)] \frac{e^{\phi_i^l}}{k_i^2(t) + q_i^2(t)} \]
\[ \pm 2y k_i^2(t)[k_i(t) - k_i(0)] e^{\phi_i^l} \]

(19a)

\[ f = W(Q_{i,1}, Q_{i,0}) \]
\[ = -4[k_i(t) - k_i(0)][-k_i(t) + k_i(0)] \frac{e^{\phi_i^l}}{k_i^2(t) + q_i^2(t)} \]
\[ + 2k_i(t)k_i(0)x + 4k_i(t)k_i(0)^2y \]
\[ + 2k_i(t)k_i(0)[k_i(t) + k_i(0)] e^{\phi_i^l}. \]

(19b)

We have obtained the N-soliton solutions of the nonisospectral KP equation by the Hirota method and Wronskian technique. Figures are given to show the shapes and motions of solutions that possess soliton characters with nonisospectral properties. We also obtain rational and mixed rational-soliton solutions. An interesting result is that we obtain a new equation with a molecular property,

\[ 4u_t + y(u_{xxx} + 6u_t + 3u^{-1}u_y) \]
\[ + 2u_t = 4u_y + 3u^{-1}u_y + 2(N - 1)u_x = 0, \]

that admits the Wronskian solution (12) where each \( \phi_j \) satisfies eq. (13a) and

\[ \phi_{j,m} = -y \phi_{j,xxx} - \frac{x}{2} \phi_{j,xxx}. \]

This new equation is found when we try to find a Wronskian solution that is equivalent to the solution in the Hirota form.

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