Exact solutions for KdV system equations hierarchy

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Abstract

Exact solutions for KdV system equations hierarchy are obtained by using the inverse scattering transform. Exact solutions of isospectral KdV hierarchy, nonisospectral KdV hierarchies and \( \tau \)-equations related to the KdV spectral problem are obtained by reduction. The interaction of two solitons is investigated.

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1. Introduction

The inverse scattering transform [1] (IST) has been widely used to solve nonlinear evolution equations [2,3]. One of the advantages of the IST is that it can be applied to a whole hierarchy of evolution equations associated with a certain spectral problem. In general, there are two sets of evolution equations, isospectral hierarchy and nonisospectral hierarchy associated to the same spectral problem. Isospectral equations often describe the solitary waves in the lossless and uniform media. The nonisospectral, resulting from a spectral problem with time-dependent spectral parameter \( \eta \) [4–7], can also be solved by the IST, and their solutions demonstrate the existence of solitary waves in a certain type of nonuniform media [6–9]. \( \tau \)-hierarchy is linear combination of isospectral hierarchy and nonisospectral hierarchy, and isospectral hierarchy and \( \tau \)-hierarchy compose two sets of symmetry with spectral problem [10].

In this paper, by using the IST, we will solve the initial-value problem for the following KdV system equation hierarchy:

\[ u_t = \alpha(t) T^{n+s-1} u_x + \beta(t) T^n (xu_x + 2u) \quad (n = 0, 1, 2, \ldots; s = 1, 2, \ldots), \]  

\[ T = \frac{\partial^2}{\partial x^2} + 4u + 2u_x \partial^{-1}, \quad \partial = \frac{\partial}{\partial x}. \]
which is associated with the following Lax pair \([4]\)

\[
\begin{align*}
\phi_{xx} + u\phi &= \lambda \phi, \\
\phi_t &= A\phi + B\phi_x,
\end{align*}
\]

(1.2a)

(1.2b)

where \(\lambda_t = \frac{4}{3}(4\lambda)^{n+1}\) and

\[
B = x(4\lambda)^n\beta(t) + z(t)(4\lambda)^{n+s-1} + 2z(t) \sum_{j=1}^{n+s-1} (4\lambda)^{n+s-1-j}\partial^{-1}T^{j-1}u_x \\
+ 2\beta(t) \sum_{j=s}^{n+s-1} (4\lambda)^{n+s-1-j}\partial^{-1}T^{s-j}(xu_x + 2u),
\]

(1.3)

\(z(t), \beta(t)\) are continuous functions with respect to \(t\).

In particular

(1) As \(z(t) = 1, \beta(t) = 0,\)

\[u_t = T^{n+s-1}u_x, \quad n + s = 1, 2, \ldots,\]

is isospectral KdV hierarchy \([11,12]\); the first two nontrivial equations are

\[u_t = K_1 \equiv u_{xxx} + 6uu_x,\]

(1.4a)

\[u_t = K_2 \equiv u_{xxxx} + 10uu_{xx} + 20u_xu_{xx} + 30u_x^2u_x,\]

(1.4b)

(2) As \(z(t) = 0, \beta(t) = 1,\)

\[u_t = T^n(xu_x + 2u),\]

(1.5a)

is nonisospectral KdV hierarchy \([4,12]\); the first two nontrivial equations are

\[u_t = xK_1 + 4u_{xx} + 8u_x^2 + 2u_x\partial^{-1}u,\]

(1.5b)

\[u_t = xK_2 + 6K_{1,xx} + 12uu_{xx} + 32u_x^3 + 2K_1\partial^{-1}u + 6u_x\partial^{-1}u_x^2,\]

(1.5c)

(3) As \(z(t) = (2s-1)t, \beta(t) = 1\)

\[u_t = (2s-1)tT^{n+s-1}u_x + T^n(xu_x + 2u),\]

(1.6a)

is \(\tau\)-hierarchy for KdV system \([10]\).

Letting \(s = 2, n = 0, \tau\)-equation is

\[u_t = 3t(u_{xxx} + 6uu_x) + xu_x + 2u.\]

(1.6b)

In this paper, for completeness, we would first briefly list some results of the direct scattering problem in Section 2. In Section 3, we derive the time evolution of the scattering data. In Section 4, we obtain reflectionless potentials for the KdV system equations hierarchy (1.1). In Section 5, the reductions are considered. Finally, in Section 6, the motions of two-soliton-like solutions for nonisospectral KdV hierarchy are investigated.

### 2. The direct scattering problem

We will list some results \([2,3]\) of the direct scattering problem related to the spectral problem (1.2a). And for convenience, we replace \(\lambda\) with \(-k^2\) in the spectral problem (1.2a).

(1) Suppose that real potential \(u(t, x)\) satisfy \(\int_{-\infty}^{+\infty} |x^ju(x)| \, dx < +\infty \) \((j = 0, 1, 2)\).

(2) There are two Jost solutions for the spectral problem (1.2a), \(\phi_1(x, k)\) and \(\phi_2(x, k)\), which are bounded for all values of \(x\) and are analytic on \(\text{Im} \, k > 0\) and continuous on \(\text{Im} \, k \geq 0\) for \(k, \phi_1(x, k)\) and \(\phi_2(x, k)\) satisfy...
the following asymptotic behaviors:
\[ \phi_1(x,k) \sim e^{ikx} \text{ as } x \to +\infty, \]  
\[ \phi_2(x,k) \sim e^{-ikx} \text{ as } x \to -\infty. \]  
(2.1a)

(3) For all real value \( k \), there are linear relationships between two Jost solutions
\[ \phi_2(x,k) = a(k)\phi_1(x,-k) + b(k)\phi_1(x,k), \]  
(2.2a)

\[ \phi_2(x,-k) = b(-k)\phi_1(x,-k) + a(-k)\phi_1(x,k), \]  
(2.2b)

where
\[ a(k) = \frac{1}{2ik} W(\phi_2(x,k),\phi_1(x,k)), \quad b(k) = \frac{1}{2ik} W(\phi_1(x,-k),\phi_2(x,k)), \]  
(2.3a,b)

and \( a(k) \) is analytic on \( \text{Im } k > 0 \) and continuous on \( \text{Im } k \geq 0 \), while \( b(k) \) is only defined on \( \text{Im } k = 0 \).

(4) The function \( a(k) \) has a finite number of simple zeros at \( k_1,k_2,\ldots,k_l \) on imaginary axis of the upper half \( k \)-plane, i.e., \( k_j = ik_j, k_j > 0, j = 1,2,\ldots,l \).

(5) Scattering data for the spectral problem (1.2a) are defined by
\[ S(t) : \left\{ \rho(k) = \frac{a(k)}{b(k)}, \quad \text{Im } k = 0, [ik_j,c_j], \quad k_j > 0 (j = 1,2,\ldots,l) \right\} \]  
(2.4)

where \( c_j^2 = -\frac{b_j}{\bar{a}(ik_j)} \) and each \( c_j \) is the so-called normalization constant of eigenfunction \( \phi_1(x,ik_j) \).

3. The time evolutions of the scattering data

In this part, we will derive the time evolutions of the scattering data.

**Lemma 3.1.** Suppose that \( \phi(x,k) \) is a solution of (1.2a), \( A \) and \( B \) satisfy the compatible condition \( \phi_{xxt} = \phi_{txx} \), i.e.,
\[ 2A_x + B_{xx} = 0, \quad u_t = -A_{xx} - 2(\lambda - u)B_x + u_xB + \lambda_t, \]
then
\[ P(x,k) = \phi_1(x,k) - A\phi(x,k) - B\phi_\bar{x}(x,k) \]  
(3.1)
solves (1.2a) as well.

**Theorem 3.1.** The related discrete scattering data in (2.4) satisfy the following time evolutions
\[ k_j(t) = \frac{1}{(k_j^2 - 2n - 4^n \int_0^t \beta(z) \, dz)^{\frac{1}{2n}}}, \]  
(3.2a)

\[ c_j(t) = c_j(0)e^{\int_{t_0}^t [4^n(2^{n+1} - s + 1)z^2 - 4^{n+1} - 4s\beta(z)] \, dz}, \]  
(3.2b)

where \( n = 0,1,2,\ldots; \quad s = 1,2,\ldots, j = 1,2,\ldots,l \), and \( \{ik_j(0), c_j(0)\} \) is the scattering data of (1.2a) as \( u(t,x) = u(0,x) \).

**Proof.** Consider the following three solutions of (1.2a) with \( k = ik_j(k_j > 0) \),
\[ P(x,ik_j) = \phi_1(x,ik_j) - A\phi(x,ik_j) - B\phi_\bar{x}(x,ik_j), \]
where \( \phi(x,ik_j) = c_j\phi_1(x,ik_j) \) is a normalized Jost function, \( c_j \) is the related normalization constant, and \( \phi_\bar{x}(x,ik_j) \) is another Jost function linearly independent of \( \phi(x,ik_j) \).

There exist two constants \( \gamma_1,\gamma_2 \) such that
\[ \phi_1(x,ik_j) - A\phi(x,ik_j) - B\phi_\bar{x}(x,ik_j) = \gamma_1\phi(x,ik_j) + \gamma_2\phi_\bar{x}(x,ik_j). \]  
(3.3)
Noting the fact of $\phi(x, ik) \to 0$ as $x \to +\infty$, it is easily to see $\gamma_2 = 0$, i.e.,
$$\phi_j(x, ik) - A\phi_j(x, ik) - B\phi_j(x, ik) = \gamma_1 \phi_j(x, ik).$$  \hfill (3.4)

To derive the value of $\gamma_1$, multiplying (3.4) by $2\phi(x, ik)$, and noting that $A = -\frac{1}{2}B$, yields
$$2\phi(x, ik)\phi_j(x, ik) + B\phi_j^2(x, ik) - 2B\phi(x, ik)\phi_j(x, ik) = 2\gamma_1 \phi_j^2(x, ik).$$  \hfill (3.5)

Further we have
$$\frac{d}{dt} \int_{-\infty}^{\infty} \phi^2(x, ik) \, dx + \int_{-\infty}^{\infty} [B_v \phi^2(x, ik) - B(\phi^2(x, ik))] \, dx = 2\gamma_1 \int_{-\infty}^{\infty} \phi^2(x, ik) \, dx.$$  \hfill (3.6)

Next, by using $\int_{-\infty}^{\infty} \phi^2(x, ik) \, dx = 1$ and $B\phi^2(x, ik) \to 0$, as $x \to \pm \infty$, we can reach
$$\gamma_1 = -\int_{-\infty}^{\infty} B(\phi^2(x, ik)) \, dx.$$  \hfill (3.7)

Then we define inner product $\langle B, (\phi^2)_x \rangle$ by $\int_{-\infty}^{\infty} B(\phi^2)_x \, dx$ and define $\delta^{-1} = \frac{1}{2}(\int_{-\infty}^{x} dx - \int_{x}^{\infty} dx)$, we have
$$\gamma_1 = -\langle B, (\phi^2)_x \rangle = -\langle z(t)(4\kappa_j)^n x, (\phi^2)_x \rangle - \sum_{j=1}^{n+1} 2x(t) \cdot (4\kappa_j)^{n+1-j} \delta^{-1} T^{j-1}(ux_x, (\phi^2)_x)$$
$$= -\langle (4\kappa_j)^n \beta(t)x, (\phi^2)_x \rangle - \sum_{j=1}^{n+1} 2\beta(t) \cdot (4\kappa_j)^{n+1-j} \delta^{-1} T^{j-1}(ux_x + 2u), (\phi^2)_x \rangle$$
$$= (n + 1)\beta(t)(2\kappa_j)^{2n},$$

where we have made use of the facts that $(\phi^2)_x$ is an eigenfunction of the operator $T$ with respect to the eigenvalue $4\kappa$, and $\delta^{-1} T = \delta + 2\delta^{-1} u + 2u\delta^{-1}$ is an anti-symmetric operator with respect to the inner product $\langle \cdot, \cdot \rangle$. Thus (3.4) becomes
$$\phi_j(x, ik) + \frac{1}{2}B_v \phi(x, ik) - B\phi(x, ik) = (n + 1)\beta(t)(2\kappa_j)^{2n} \phi(x, ik).$$  \hfill (3.8)

Setting $x \to +\infty$ in (3.8) yields
$$c_{j,t} - n\kappa_j c_j + \frac{1}{2}\beta(t)(2\kappa_j)^{2n} c_j + x\beta(t)\kappa_j(2\kappa_j)^{2n} c_j + \kappa_j x(t)(2\kappa_j)^{2n+2n-2} = (n + 1)\beta(t)(2\kappa_j)^{2n} c_j,$$  \hfill (3.9)

which further means
$$\kappa_j (t) = 4^n \beta(t) \kappa_j^{2^{n+1}}(t), \quad c_{j,t}(t) = (4^n(n + 1)\beta(t)\kappa_j^{2^n}(t) - 4^{n+1-1} z(t)\kappa_j^{2^{n+1-1}})c_j(t).$$

Thus we complete the proof. \hfill \Box

4. I-soliton-like solution for KdV system equation hierarchy

In this section, we will derive out the reflectionless potentials of the KdV system equation hierarchy (1.1).

**Theorem 4.1** (Ablowitz and Clarkson [2]). For the scattering data (2.4), we define
$$F(t, x) = F_v(t, x) + F_d(t, x),$$  \hfill (4.1a)

where
$$F_v(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(t, k(t)) e^{ik(t)x} \, dk, \quad F_d(t, x) = \sum_{j=1}^{l} c_j^2(t) e^{-\kappa_j(t)x}. \hfill (4.1b,c)$$

Then we have
$$u(t, x) = 2 \frac{d}{dx} K(t, x),$$  \hfill (4.2)
where $K(t,x,y)$ solves the Gel’fand–Levitan–Marchenko integral equation

$$K(t,x,y) + F(t,x + y) + \int_x^\infty F(t,z + y)K(t,x,z)\,dz = 0.$$  (4.3)

Particularly, when the reflection coefficient $\rho(t,k(t)) = 0$, (4.3) becomes

$$K(t,x,y) + \sum_{j=1}^l c_j^2(t)e^{-k_j(t)(x+y)} + \sum_{j=1}^l e_j^2(t)e^{-\kappa_j(t)y} \int_x^\infty K(t,x,z)e^{-\kappa_j(t)z}\,dz = 0.$$  (4.4)

To obtain solutions, we first set

$$K(t,x,y) = \sum_{n=1}^l c_n(t)\delta_n(x)e^{-\kappa_n(t)y}.$$  (4.5)

Then, substituting (4.5) into (4.4), we obtain

$$K(t,x,x) = \frac{d}{dx} \ln \det D(t,x),$$  (4.6)

where $D(t,x) = (d(t,x))_{l \times l}$, $d(t) = \delta(t) + \frac{1}{\kappa(t) + \kappa(t)} c_i(t)c_j(t)e^{-(\kappa_i(t) + \kappa_j(t))x}$. Therefore, the l-soliton solution of the KdV system equation hierarchy (1.1) is given by

$$u(t,x) = 2 \frac{d^2}{dx^2} \ln \det D(t,x).$$  (4.7)

5. Reductions

In this section, we consider some reductions of the KdV system equation hierarchy.

Firstly, we investigate the isospectral KdV hierarchy. As $\beta(t) = 0$, $\alpha(t) = 1$, Eq. (1.1) becomes isospectral KdV hierarchy (1.4a), it is easy to obtain the following relations of scattering from (3.2)

$$k_j(t) = k_j(0), \quad c_j(t) = c_j(0)e^{-(2k_j)^2t + \kappa_j t} \quad (j = 1, 2, \ldots).$$  (5.1)

1. As $n + s - 1 = 1$, $l = 1$, Substituting (5.1) into (4.7), one-soliton solution for Eq. (1.4b) is given by

$$u = 2k^2 \text{sech}^2 \left( \frac{2kx + 8k^3t + \frac{1}{2} \ln \frac{c^2}{2k}}{2} \right),$$  (5.2)

where $c$, $k$ are constant.

2. As $n + s - 1 = 1$, $l = 2$, Substituting (5.1) into (4.7), two-soliton solution for Eq. (1.4b) is given by

$$u = \frac{2c_1^2 k_1 e^{\xi_1} + 2c_2^2 k_2 e^{\xi_2} + \frac{2c_1^2 c_2^2}{k_1 k_2} (k_1 - k_2)^2 e^{\xi_1 + \xi_2} + c_1^4 c_2^2 k_2^2 (k_1 - k_2)^2 e^{\xi_1 + \xi_2} + c_1^2 c_2^4 k_1^2 (k_1 - k_2)^2 e^{\xi_1 + \xi_2}}{(1 + \frac{c_1^2}{2k_1} e^{\xi_1} + \frac{c_2^2}{2k_2} e^{\xi_2} + \frac{c_1^2 c_2^2}{4k_1 k_2 (k_1 + k_2)} e^{\xi_1 + \xi_2})^2},$$  (5.3)

where $\xi_j = -2k_j x - 8k_j^3 t$, $c_i$, $k_i$ $(i = 1, 2)$ are constants. The above results are consistent with Ref. [2].

(3) As $n + s - 1 = 2$, $l = 1$, Substituting (5.1) into (4.7), one-soliton solution for Eq. (1.4c) is given by

$$u = 2k^2 \text{sech}^2 \left( kx + 18k^5t + \frac{1}{4} \ln \frac{c^2}{2k} \right),$$  (5.4)

where $c$, $k$ are constant.

Secondly, we consider the nonisospectral KdV hierarchy. As $\beta(t) = 1$, $\alpha(t) = 0$, Eq. (1.1) becomes nonisospectral KdV hierarchy (1.5a).

From (3.2), we give the following relations of scattering

$$\kappa_j(t) = \frac{1}{(\kappa_j^{-2n}(0) - 2n \cdot 4^n t)^{\frac{1}{n}}},$$  (5.5a)
\[
c_j(t) = c_j(0) \left( \frac{\kappa_j^{-2n}(0)}{\kappa_j^{-2n}(0) - 2n \cdot 4^n t} \right)^{\frac{2n+1}{4n}} \quad (j = 1, 2, \ldots, l). \tag{5.5b}
\]

(1) As \(n = 1, 2, l = 1\), Substituting (5.5) into (4.7), one-soliton solutions for Eq. (1.5b,c) are given by

\[
u(t, x) = \frac{1}{2(c - 2t)} \text{sech}^2 \left( \frac{1}{2} \ln(c - 2t) + \frac{x}{2\sqrt{c - 2t}} \right),
\tag{5.6}
\]

\[
u(t, x) = \frac{1}{2\sqrt{c - 4t}} \text{sech}^2 \left( \frac{1}{2} \ln(c - 4t) + \frac{x}{2\sqrt{c - 4t}} \right),
\tag{5.7}
\]

where \(c\) is constant.

(2) As \(n = 1, 2, l = 2\), Substituting (5.5) into (4.7) give two-soliton solutions for (1.5b,c)

\[
u(t, x) = 2 \left( \frac{e^{\theta_1} + e^{\theta_2}}{c_1 - 2t} + \frac{e^{\theta_1} + e^{\theta_2}}{c_2 - 2t} + 2 \left( \frac{1}{\sqrt{c_1 - 2t}} - \frac{1}{\sqrt{c_2 - 2t}} \right)^2 \frac{A e^{\theta_1+2\theta_2}}{c_1 - 2t} + \frac{A e^{\theta_1+2\theta_2}}{c_2 - 2t} \right),
\tag{5.8}
\]

where \(c_1\) and \(c_2\) are constants, \(\theta_j = -\ln(c_1 - 2t) - \frac{x}{\sqrt{c_1 - 2t}} (j = 1, 2), A = \left( \frac{\sqrt{c_1+2t} - \sqrt{c_2+2t}}{\sqrt{c_1+2t} + \sqrt{c_2+2t}} \right)^2 \).

\[
u(t, x) = \left( \frac{e^{\theta_1} + e^{\theta_2}}{\sqrt{c_1 - 2t} - 2t} + \frac{e^{\theta_1} + e^{\theta_2}}{\sqrt{c_2 - 2t} - 2t} + 2 \left( \frac{1}{\sqrt{c_1 - 4t}} - \frac{1}{\sqrt{c_2 - 4t}} \right)^2 \frac{A e^{\theta_1+2\theta_2}}{\sqrt{c_1 - 4t}} + \frac{A e^{\theta_1+2\theta_2}}{\sqrt{c_2 - 4t}} \right),
\tag{5.9}
\]

where \(c_1\) and \(c_2\) are constants, \(\theta_j = -\ln(c_j - 4t) - \frac{x}{\sqrt{c_j - 4t}} (j = 1, 2), A = \left( \frac{\sqrt{c_1+4t} - \sqrt{c_2+4t}}{\sqrt{c_1+4t} + \sqrt{c_2+4t}} \right)^2 \).

Finally, we investigate the \(\tau\)-equations related to the KdV spectral problem. As \(\beta(t) = 1\) and \(z(t) = (2s - 1)t\), Eq. (1.1) becomes \(\tau\)-equations for KdV spectral problem (1.6a). It is easy to give the following relations of scattering from (3.2)

\[
\kappa_j(t) = \frac{1}{(\kappa_j^{-2n}(0) - 2n \cdot 4^n t)^{1/2n}}, \tag{5.10a}
\]

\[
c_j(t) = c_j(0) \left( \frac{\kappa_j^{-2n}(0)}{\kappa_j^{-2n}(0) - 2n \cdot 4^n t} \right)^{\frac{2n+1}{4n}} \quad (j = 1, 2, \ldots, l), \tag{5.10b}
\]

\[
\Delta = e^{-\frac{2n+1}{4n} t} (\kappa_j^{-2n}(0) - 2n \cdot 4^n t)^{1/2n}. \tag{5.11}
\]

In particularly, as \(n = 0, s = 2, l = 1\), one-soliton for \(\tau\)-equation (1.6b) is

\[
u = 2k^2 e^{2t} \text{sech}^2 \left( k x e^t + \frac{4}{3} k^3 e^{3t} (3t - 1) - \frac{4}{3} k^3 - \frac{1}{2} \ln \frac{c^2}{2k} \right),
\tag{5.11}
\]

where \(c\) and \(k\) are constants.

These solutions can also be verified by direct substitutions.

6. The motions of 2-soliton-like solutions for nonisospectral KdV hierarchy

Although the hierarchy (1.5a) is equations with nonisospectral characters, and the width and amplitude of the wave vary with time, the wave still exhibits scattering characters similar to soliton interactions [3].

In this section, for convenience, we replace \(t\) with \(-t\). Then the hierarchy (1.5a) becomes

\[
u_t + T^n (nu_x + 2u) = 0. \tag{6.1}
\]

In this case, for \(n = 1\) and \(l = 2\), using the same method given in Ref. [9], we decompose the 2-soliton-like solution (5.8) as

\[
u(t, x) = u_1(t, x) + u_2(t, x), \tag{6.2}
\]
where
\[ u_1(t, x) = \frac{1}{2(c_1 + 2t)} p_1(\theta_2) \text{sech}^2(\theta_1 + G(\theta_2)), \quad u_2(t, x) = \frac{1}{2(c_2 + 2t)} p_2(\theta_1) \text{sech}^2(\theta_2 + G(\theta_1)), \] (6.3)
and
\[ p_i(\theta_j) = \frac{1 + B_i e^{2\theta_j} + Ae^{4\theta_j}}{1 + (1 + A)e^{2\theta_j} + Ae^{4\theta_j}} \quad (i, j = 1, 2, i \neq j), \]
\[ A = \left( \frac{\sqrt{c_1 + 2t} - \sqrt{c_2 + 2t}}{\sqrt{c_1 + 2t} + \sqrt{c_2 + 2t}} \right)^2, \quad B_1 = -B_2 = -2A^2, \]
\[ G(\theta_j) = \frac{1}{2} \ln \left( \frac{1 + Ae^{2\theta_j}}{1 + e^{2\theta_j}} \right), \quad \theta_j = -\frac{1}{2} \left[ \ln(c_j + 2t) + \frac{x}{\sqrt{c_j + 2t}} \right] \quad (j = 1, 2). \]

Fig. 1. The shape and motion of the 2-soliton solution for \( c_1 = 4, c_2 = 8 \). (a) \( t = -1.99 \); (b) \( t = 0 \); (c) \( t = 7 \); (d) \( t = 12000 \).
Since $e^{\sqrt{t}}$ rapidly tends to 1 as $t \to +\infty$, we have
\[ p_i(\theta_j) \approx \frac{(c_j + 2t + \sqrt{A})^2}{(c_j + 2t)^2 + (c_j + 2t)(1 + A) + A}, \quad G(\theta_j) \approx \frac{1}{2} \ln \frac{c_j + 2t + A}{c_j + 2t + 1} \quad (t \to +\infty). \]

Thus, for the fixed $\theta_1$, as $t \to +\infty$, we have
\[ u_1(t, x) \approx \frac{1}{2(1 + A) \sqrt{c_1 + 2t + 1}} \ln \frac{c_1 + 2t + A}{c_1 + 2t + 1}; \]
for the fixed $\theta_2$, as $t \to +\infty$, we have
\[ u_2(t, x) \approx \frac{1}{2(1 + A) \sqrt{c_2 + 2t + 1}} \ln \frac{c_2 + 2t + A}{c_2 + 2t + 1}. \]

Assume $c_1 < c_2$, it is shown that solitary wave $u_2$ rises earlier than solitary wave $u_1$. When $u_2$ travels leftwards, it collides with the later risen wave $u_1$ and then separates off each other; the interaction of solitary wave $u_1$ and $u_2$ results in the change of the amplitudes and phase shifts of $u_1$ and $u_2$.

Fig. 1 shows the process that solitary wave $u_2$ collide with solitary wave $u_1$ and separates off as time-variable $t$ from $-2$ to $\infty$.

To sum up, by using the standard IST procedure, we have derived out the reflectionless potentials for the KdV system equation hierarchy (1.1) and investigate some reductions. Exact solutions for the isospectral KdV hierarchy, the nonisospectral KdV hierarchy and the $\tau$-equation hierarchy related to KdV spectral problem. We also discussed the scattering characters of the solutions for nonisospectral KdV equation. Although the nonisospectral KdV hierarchy (1.5a) is equations with nonisospectral characters, and the width and amplitude of the wave vary with time, the wave still exhibits scattering characters similar to soliton interactions.

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References