Resonances of Line Solitons in a Non-isospectral Kadomtsev–Petviashvili Equation

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It is well known that the Kadomtsev–Petviashvili (KP) equation provides line solitons in shallow water and these solitons can be of resonance.1,2 Recently, a non-isospectral KP equation was investigated.3 This equation reads

\[ 4u_t + y(tu_{xxx} + 6uu_x + 3\alpha^{-1}u_y) + 2xu_y + 4\alpha^{-1}u_y = 0, \]  

(1)

which is integrable with the Lax pair

\[ \begin{align*}
\phi_t &= \phi_{xx} + 2\alpha \phi, \\
\phi_x &= y \left[ \phi_{xxx} + 3\alpha \phi_x + \frac{3}{2} (\alpha^{-1} u_x + u_y) \right] + \frac{1}{2} x (\phi_{xx} + 2\alpha \phi) + \frac{1}{2} (\alpha^{-1} u) \phi.
\end{align*} \]

(2a)

(2b)

By the dependent variable transformation

\[ u = 2(\ln P)_{xx}, \]

(3)

eq (1) admits a bilinear form3

\[ 4D_t D_y f \cdot f + y(D_x^3 f \cdot f + 3D_x^2 f \cdot f) + 2xD_t D_y f \cdot f + 4ff_x = 0, \]

(4)

which has been solved by3

\[ f = \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu_j \theta_j + \sum_{1 \leq j l} \mu_j \mu_l A_{jl} \right), \]

(5a)

with

\[ \begin{align*}
\theta_1 &= K_1(t)[x + P_1(t)y] + \theta_1^{(0)}, \\
K_{ij}(t) &= -\frac{1}{2} K_i(t) P_j(t), \\
P_{ij}(t) &= -\frac{1}{4} K_i^2(t) - \frac{1}{4} P_j^2(t), \\
K_f(t) &= \frac{8c_i}{4c_j - t^2}, \\
P_f(t) &= \frac{-4t}{4c_j - t^2}, \\
e^{ix} &= \left[ \frac{|K_f(t) - K_i(t)|^2}{|K_f(t) + K_i(t)|^2} - \frac{|P_f(t) - P_i(t)|^2}{|P_f(t) + P_i(t)|^2} \right],
\end{align*} \]

(5b)

(5c)

(5d)

(5e)

where \( \theta_1^{(0)} \) and \( c_i \) are all real constants, the sum over \( \mu = 0, 1 \) refers to each of \( \mu_j = 0, 1 (j = 1, 2, \ldots, N) \). In eq. (4), \( D \) is the Hirota’s bilinear operator4 defined as

\[ D_x^m D_y^n f \cdot g = (\partial_x - \partial_y)^m (\partial_x - \partial_y)^n f(t, x) g(t, x)' \big|_{x=t, x'=t}. \]

(6)

In our short note, beyond (5c) we give the following new solutions to (5c),

\[ K_f(t) = \frac{2}{t - 2c_j} - \frac{2}{t + 2d_i}, \]

(7a)

\[ P_f(t) = \frac{2}{t - 2c_j} + \frac{2}{t + 2d_i}, \]

(7b)

and in the light of this, \( e^{ix} \) simply is

\[ e^{ix} = \left( \frac{c_j - c_i}{d_j - d_i} \right)^{d_j + d_i}. \]

(7c)

where \( c_j \) and \( d_i \) are all real constants. Thus, a more general solution to (4) can be given by (5a) and (5b) with (7a) and (7b).

The resonance of line solitons will appear when \( e^{ix} = 0 \) in non-degenerate multi-soliton case. Let us consider a 2-soliton solution which is described through

\[ f = 1 + e^{ix} + e^{ix}. \]

(8)

Now let us investigate the asymptotic behaviors of such two solitons by means of asymptotic approach as did in ref. 2.

We consider the following three cases. First, when

\[ t < 2c_1 \quad \text{or} \quad t > 2c_2, \]

(10)

we introduce the coordinate frame \([X = x + P_1(t)y, y]\) and under which we have

\[ \theta_1 = K_1(t)X + \theta_1^{(0)}, \]

\[ \theta_2 = K_2(t)X + K_2(t)P_2(t) - P_1(t)y + \theta_2^{(0)}. \]

The condition (10) guarantees \( K_2(t)[P_2(t) - P_1(t)]y + \theta_2^{(0)} \) always positive. Now letting the frame co-moves with \( \theta_1 \)-soliton, i.e., keeping \( X \) constant, we have \( \theta_2 \rightarrow \pm \infty, y \rightarrow \pm \infty \), which further suggests

\[ u \rightarrow \begin{cases} 
\frac{K_1(t)}{2} \text{sech}^2 \frac{\theta_1}{2}, & y \rightarrow -\infty, \\
0, & y \rightarrow +\infty.
\end{cases} \]

(11)

Similarly, in the coordinate frame \([Y = x + P_2(t)y, y]\), if \( Y \) stays constant, we have

\[ u \rightarrow \begin{cases} 
\frac{K_2(t)}{2} \text{sech}^2 \frac{\theta_2}{2}, & y \rightarrow -\infty, \\
0, & y \rightarrow +\infty.
\end{cases} \]

(12)

In the coordinate frame \([Z = x - [P_1(t) + P_2(t)]y, y]\), if \( Z \) stays constant,
Fig. 1. The density plots of two soliton resonances for the non-isospectral KP given by (3) with (9). (a) \( t = -5.4, c_1 = -2, c_2 = 1.6, d_1 = d_2 = 1 \);
(b) \( t = -0.2, c_1 = -2.4, c_2 = 1.8, d_1 = d_2 = 1 \); (c) \( t = -3.6, c_1 = -2.4, c_2 = 1.8, d_1 = d_2 = 1 \), where \( \theta_1^0 = \theta_2^0 = 0 \).

Let us rewrite the above asymptotic behaviors in a simple form

\[
\begin{align*}
0, & \quad y \to -\infty, \\
\frac{[K_1(t) - K_2(t)]^2}{2} \text{sech}^2 \left( \frac{\theta_1 - \theta_2}{2} \right), & \quad y \to +\infty.
\end{align*}
\]

(13)

Let us rewrite the above asymptotic behaviors in a simple form

\[
\begin{align*}
\frac{K_1(t)^2}{2} \text{sech}^2 \frac{\theta_1}{2}, & \quad y \to -\infty, \\
\frac{K_2(t)^2}{2} \text{sech}^2 \frac{\theta_2}{2}, & \quad y \to -\infty, \\
\frac{[K_1(t) - K_2(t)]^2}{2} \text{sech}^2 \left( \frac{\theta_1 - \theta_2}{2} \right), & \quad y \to +\infty.
\end{align*}
\]

(14)

For the other two cases, when \(-2d_1 < t < 2c_2\), we have

\[
\begin{align*}
\frac{K_1(t)^2}{2} \text{sech}^2 \frac{\theta_1}{2}, & \quad y \to -\infty, \\
\frac{K_2(t)^2}{2} \text{sech}^2 \frac{\theta_2}{2}, & \quad y \to +\infty, \\
\frac{[K_1(t) - K_2(t)]^2}{2} \text{sech}^2 \left( \frac{\theta_1 - \theta_2}{2} \right), & \quad y \to -\infty.
\end{align*}
\]

(15)

when \(2c_1 < t < -2d_1\), we have

\[
\begin{align*}
\frac{K_1(t)^2}{2} \text{sech}^2 \frac{\theta_1}{2}, & \quad y \to +\infty, \\
\frac{K_2(t)^2}{2} \text{sech}^2 \frac{\theta_2}{2}, & \quad y \to -\infty, \\
\frac{[K_1(t) - K_2(t)]^2}{2} \text{sech}^2 \left( \frac{\theta_1 - \theta_2}{2} \right), & \quad y \to -\infty.
\end{align*}
\]

(16)

We depict the above three cases in Fig. 1.

To sum up, we have investigated the resonance of two line solitons of the non-isospectral KP equation by asymptotic analysis. Since the amplitude and slope \((dy/dx)\) of each line soliton are time-dependent, such resonance might describe a special behavior of wave motions in some kind of non-uniform media. Meanwhile, in the light of (7a) we note that the soliton resonant solutions can become singular at a finite time and there will be a blowup at \(t = 2c_j\) or \(t = -2d_j\).

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