Solving the KdV Equation Under Bargmann Constraint via Bilinear Approach\* 

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Abstract In this paper we consider exact solutions to the KdV equation under the Bargmann constraint. Solutions expressed through exponential polynomials and Wronskians are derived by bilinear approach through solving the Lax pair under the Bargmann constraint. It is also shown that the potential \( u \) in the stationary Schrödinger equation can be a summation of squares of wave functions from bilinear point of view.

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1 Introduction 
It is well known that under the Bargmann constraint many soliton equations can be finite dimensional completely integrable systems.\textsuperscript{[1–4]} Meanwhile, the constraint provides a starting point to get solutions by algebraic geometry approach.\textsuperscript{[5–6]} In some sense the Bargmann constraint is considered to be nature. In fact, taking the KdV equation as an example, the constraint can be derived from the Gel’fand–Levitan–Marchenko (GLM) equation in the Inverse Scattering Transform.\textsuperscript{[7–8]} This suggests that the Lax pair under the Bargmann constraint might be solvable and yields exact solutions to the related evolution equations.

In the letter we derive exact solutions to the KdV equation by solving its Lax pair under the Bargmann constraint. We employ bilinear approach to achieve that. By some transformations the Lax pair and Bargmann constraint are transformed into bilinear equations and they can further be solved by Hirota method and Wronskian technique. Besides, we show that the potential \( u \) in the stationary Schrödinger equation can be a summation of squares of wave functions and the wave functions can precisely be described Wronskians. That means the Bargmann constraint is also reasonable from bilinear point of view. This was not reported before.

The letter is organized as follows. In Sec. 2 we recall the KdV equation and the Bargmann constraint. In Sec. 3 we solve the Lax pair and the Bargmann constraint by bilinear approach. Section 4 consists of conclusions and remarks.

2 The KdV Equation and Its Bargmann Constraint 
We first recall the KdV equation and its Bargmann constraint.

The KdV equation is

\[ u_t - 6uu_x - u_{xxx} = 0, \] (1)

with its Lax pair

\[ \varphi_{xx} = (\lambda - u)\varphi, \] (2a)

\[ \varphi_t = \varphi_{xxx} + 3(\lambda + u)\varphi_x. \] (2b)

For the KdV equation, the so-called Bargmann constraint is referred to as

\[ u = 2 \sum_{i=1}^{N} \varphi_i^2, \] (3)

where each \( \varphi_i \) solves the Lax pair (2) with \( \lambda = \lambda_i \), i.e., satisfying

\[ \varphi_{i,xx} = (\lambda_i - u)\varphi_i, \]

\[ \varphi_{i,t} = \varphi_{i,xxx} + 3(\lambda_i + u)\varphi_{i,x}, \quad i = 1, 2, \ldots, N. \] (4)

For convenience, in the following we call (3) and (4) the Bargmann-Lax system. It is well known that starting from the Lax pair (2) or (4) one can solve the KdV equation by the Inverse Scattering Transform\textsuperscript{[7–8]} and the Bargmann constraint (3) is natural and can also be derived from the GLM equation. In this sense, the Bargmann-Lax system (3) and (4) can provide exact solutions for the KdV equation as well. In the paper we consider the system by means of the bilinear approach.

3 Bilinear Approach for Bargmann-Lax System 
In the section we first transform the Bargmann-Lax system into bilinear form and then solve it by Hirota method\textsuperscript{[9–10]} and Wronskian technique.\textsuperscript{[11]}

3.1 Hirota Method 
By the transformation

\[ u = 2(\ln f)_{xx}, \] (5a)

\[ \varphi_i = \frac{g_i}{f} \] (5b)
for $i = 1, 2, \ldots, N$, we rewrite (3) and (4) into the following bilinear form

$$D^2_x f f = 2 \sum_{i=1}^{N} g_i^2,$$  \hspace{1cm} (6a)$$
$$D^2_x g_i f = \lambda_i g_i f,$$  \hspace{1cm} (6b)$$
$$(D_t - D^2_x) g_i f - 3 \lambda_i D_x g_i f = 0$$  \hspace{1cm} (6c)$$

for $i = 1, 2, \ldots, N$, where $D$ is the well-known Hirota’s bilinear operator defined by\(^{(10)}\)

$$D^n_x D^n_y a(t, x) b(t, x) = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t + s, x + y) \times b(t - s, x - y) \big|_{s=0, y=0}, \quad m, n = 1, 2, \ldots$$

Then, inserting the expansion

$$f = 1 + f^{(2)} \varepsilon^2 + f^{(4)} \varepsilon^4 + \cdots,$$

$$g_i = g_i^{(1)} \varepsilon + g_i^{(3)} \varepsilon^3 + g_i^{(5)} \varepsilon^5 + \cdots$$  \hspace{1cm} (7)$$

into Eq. (6), we have

$$f^{(2)}_{xx} = \sum_{i=1}^{N} g_i^{(1)}^2, \quad 2f^{(4)}_{xx} = -D^2_x f^{(2)} f^{(2)} + 4 \sum_{i=1}^{N} g_i^{(1)} g_i^{(3)},$$

$$f^{(6)}_{xx} = -D^2_x f^{(2)} f^{(4)} + \sum_{i=1}^{N} (g_i^{(3)})^2 + 2 g_i^{(1)} g_i^{(5)},$$

$$\vdots$$  \hspace{1cm} (8)$$

and for each $i$,

$$g_i^{(1)}_{i,xx} = \lambda_i g_i^{(1)}$$

$$g_i^{(3)}_{i,xxx} = -D^2_x g_i^{(1)} f^{(2)} + \lambda_i(g_i^{(3)} + g_i^{(1)} f^{(2)}),$$

$$g_i^{(5)}_{i,xxxx} = -D^2_x (g_i^{(3)} f^{(2)} + g_i^{(1)} f^{(4)}) + \lambda_i(g_i^{(5)} + g_i^{(3)} f^{(2)} + g_i^{(1)} f^{(4)}),$$

$$\vdots$$  \hspace{1cm} (9)$$

In order to derive $N$-solitons of the KdV equation from Eq. (6), we take $\lambda_i = k_i^2$ and

$$g_{i,t}^{(1)} = 2k_i e^{\xi_i}, \quad \xi_i = k_i x + \omega_i t + \xi_i^{(0)},$$  \hspace{1cm} (10)$$

for $i = 1, 2, \ldots, N$ where $k_i, \omega_i$, and $e^{\xi_i^{(0)}}$ are all real constants.

When $N = 1$, employing the standard Hirota’s approach, we find that Eq. (7) can be truncated by taking

$$\omega_1 = 4k_1^3, \quad f^{(2)} = e^{2\xi_1},$$

$$g_1^{(2k-1)} = f^{(2k)} = 0, \quad k = 2, 3, \ldots$$

Then from Eq. (5a) we can obtain 1-soliton solution of the KdV equation as

$$u = k_1^2 \text{sech}^2 \xi_1,$$  \hspace{1cm} (11)$$

where we have taken $\varepsilon = 1$ in Eq. (5a).

Similarly, when $N = 2$, we have truncated solutions of Eq. (6) as

$$f = 1 + e^{2\xi_1} + e^{2\xi_2} + e^{2(\xi_1 + \xi_2 + A_{12})},$$

$$g_1 = 2k_1 e^{\xi_1} + 2k_2 e^{\xi_1 + \xi_2 + A_{12}},$$

$$g_2 = 2k_2 e^{\xi_1} + 2k_1 e^{\xi_1 + \xi_2 + A_{12}},$$

with $\omega_i = 4k_i^3$ and $e^{A_{ij}} = (k_i - k_j)/(k_i + k_j)$ for $i, j = 1, 2$. Here we also have taken $\varepsilon = 1$ in Eq. (7), and the 2-soliton solution is provided by Eq. (5).

We can continue the procedure to get 3-solitons and 4-solitons. For the general $N$, we have

$$f = \sum_{\mu=0}^{N} \exp\left(\sum_{j=1}^{N} 2\mu_j \xi_j + 2 \sum_{1 \leq j < i}^{N} \mu_j \mu_i A_{ji}\right), \quad g^{(i)} = 2k_i e^{\xi_i} \left\{ \sum_{\mu=0}^{N} \exp\left[ \sum_{i \neq j}^{N} 2\mu_j \left( \xi_j + \frac{1}{2} \mu_j A_{ij}\right) + 2 \sum_{1 \leq j < i, i \neq j}^{N} \mu_j \mu_i A_{ji}\right] \right\},$$

$$\omega_i = 4k_i^3, \quad \lambda_i = k_i^2, \quad e^{A_{ij}} = \frac{k_i - k_j}{k_i + k_j}, \quad (i, j = 1, 2, \ldots, N).$$  \hspace{1cm} (12)$$

### 3.2 Solutions in Wronskian Form

An $N \times N$ Wronskian is defined by

$$W = \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix} = |\phi, \partial \phi, \ldots, \partial^{N-1} \phi| = |0, 1, 2, \ldots, N - 1| = |N - 1|,$$  \hspace{1cm} (13)$$

where $\phi^{(s)} = \partial^s \phi / \partial x^s$, $\partial = \partial / \partial x$, and $\phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$. In general, to achieve Wronskian verifications we need the following determinantal identity,

$$|M, a, b| M, c, d| - |M, a, c| M, b, d| + |M, a, d| M, b, c| = 0,$$  \hspace{1cm} (14)$$

where $M, a, b| M, c, d|$ is a shorthand for $M(a, b, c, d)$.
where $M$ is an $N \times (N-2)$ matrix and $a$, $b$, $c$, and $d$ are $N$-order column vectors.

For the solutions in Wronskian form to the bilinear Bargmann-Lax system (6), we have the following result.

**Theorem 1** The Wronskians

$$f = |\hat{N}-1|, \quad g_j = 2k_j \sqrt{(-1)^{N-j} |a_j b_j| \prod_{i=1, i \neq j}^N (k_i^2 - k_j^2) |\hat{N}-2, \tau_j|}, \quad (j = 1, 2, \ldots, N), \quad (16)$$

solve the bilinear Bargmann-Lax system (6) provided the Wronskian entries satisfy

$$\phi_j, xx = k^2 \phi_j, \quad (17a)$$

$$\phi_j, t = 4\phi_j, xxx, \quad (17b)$$

where $\tau_i = (\delta_1, \delta_2, \ldots, \delta_N)^T$, and $a_j$, $b_j$ are real nonzero constants and we set $0 < k_1 < k_2 < \ldots < k_N$ without loss of generality.

Noting that Eqs. (6b) and (6c) in the bilinear Bargmann-Lax system are nothing but the bilinear BT of the KdV equation, and Wronskian solutions for the BT have been discussed in Ref. [14], so we only need to focus on Eq. (6a). Before we give a detailed proof, let us first consider some lemmas.

**Lemma 1**

$$\sum_{j=1}^N \left| \alpha_1, \ldots, \alpha_{i-1}, b \alpha_i, \alpha_{i+1}, \ldots, \alpha_N \right| = \left| \sum_{i} b_i \right| |A|, \quad (18)$$

where $|A| = |\alpha_1, \alpha_2, \ldots, \alpha_N|$ with $N$-order column vector $\alpha_j = (\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{Nj})^T$, and $ba_j$ denotes $(b_1 \alpha_{1j}, b_2 \alpha_{2j}, \ldots, b_N \alpha_{Nj})^T$.

**Lemma 2** Suppose that $f = |\hat{N}-1|$ and the entries $\{\phi_j\}$ satisfy Eq. (17a). Then we have

$$f_{xx} f - f_x^2 = 2|\hat{N}-2, N+1| |\hat{N}-1| - \left( \sum_{i=1}^N k_i^2 \right) |\hat{N}-1|^2 - |\hat{N}-2, N|^2. \quad (19)$$

**Proof** First,

$$f_{xx} f - f_x^2 = \left( |\hat{N}-2, N+1| + |\hat{N}-3, N-1, N| \right) \times |\hat{N}-1| - |\hat{N}-2, N|^2. \quad (20)$$

Meanwhile, by means of Lemma 2 and taking $b_i = k_i^2$ and $|A| = f$ in Eq. (18), we have

$$|\hat{N}-3, N-1, N|$$

Then, inserting Eq. (21) into Eq. (20) yields Eq. (19).

**Lemma 3**

$$\prod_{i=1, i \neq l}^N (k_i^2 - k_l^2) = \sum_{m=0}^{N-1} (-1)^m k_i^{2(N-m-1)} \times \left( \sum_{i_1 < i_2 < \cdots < i_m \neq l} k_{i_1}^2 k_{i_2}^2 \cdots k_{i_m}^2 \right). \quad (22)$$

**Lemma 4**

Suppose that $\phi_j$ satisfies Eq. (17), then

$$C_j = k_j^2 \phi_j^2 - (\phi_j, x)^2 \quad (23)$$

is a constant.

In fact, in the light of Eq. (17), it is easy to show that

$$\frac{\partial C_j}{\partial x} = \frac{\partial C_j}{\partial t} = 0. \quad (24)$$

The constant $C_j$ can be determined as the following. The general real solution of Eq. (17) is

$$\phi_j = a_je^{\xi_j} + b_je^{-\xi_j}, \quad \xi_j = k_jx + 4k_j^2t + \xi_j^{(0)}, \quad (24)$$

with arbitrary real constants $a_j$, $b_j$, $k_j$, and $e^{\xi_j^{(0)}}$. Substituting Eq. (43) into Eq. (23) yields

$$C_j = k_j^2 \phi_j^2 - (\phi_j, x)^2 = 4k_j^2 a_j b_j. \quad (25)$$

**Lemma 5** Suppose that $A_{iN}$ is the cofactor of $f = |\hat{N}-1|$ with respect to the $j$-th row and $N$-th column, and the entries $\{\phi_j\}$ satisfy Eq. (17a). We define

$$P[m] = \sum_{i_1, i_2, \ldots, i_{m+2} = 1 \atop i_1 \neq i_2 \ldots \neq i_{m+2}} (\prod_{i=1}^{m} k_{i_{i+1}}^2) \phi^{(N-m)}_{i_{m+1}} \phi^{(N-m)}_{i_{m+2}} A_{i_{m+1}N} A_{i_{m+2}N}, \quad (1 \leq m \leq N-2),$$

$$Q[m] = \sum_{i_1, i_2, \ldots, i_{m+2} = 1 \atop i_1 \neq i_2 \ldots \neq i_{m+2}} (\prod_{i=1}^{m} k_{i_{i+1}}^2) \phi^{(N-m-1)}_{i_{m+1}} \phi^{(N-m-1)}_{i_{m+2}} A_{i_{m+1}N} A_{i_{m+2}N}, \quad (1 \leq m \leq N-2).$$
\[ S[m] = \sum_{i_1, i_2, \ldots, i_m+1}^{N} \left( \prod_{s=1}^{m} k_{i_s}^2 \right) \left( \phi_{i_{m+1}}^{(m)} \right)^2 A_{i_{m+1}N}^2, \quad (1 \leq m \leq N - 1), \]

\[ R[m] = \sum_{i_1, i_2, \ldots, i_m+1}^{N} \left( \prod_{s=1}^{m} k_{i_s}^2 \right) \left( \phi_{i_{m+1}}^{(m-1)} \phi_{i_{m+1}}^{(m+1)} \right) A_{i_{m+1}N}^2, \quad (1 \leq m \leq N - 1), \]  

and specially,

\[ P[0] = \sum_{i_1, i_2=1}^{N} \phi_{i_1}^{(N)} \phi_{i_2}^{(N)} A_{i_1N} A_{i_2N}, \quad (27a) \]

\[ Q[0] = \sum_{i_1, i_2=1}^{N} \phi_{i_1}^{(N-1)} \phi_{i_2}^{(N+1)} A_{i_1N} A_{i_2N}, \quad (27b) \]

\[ S[0] = \sum_{i=1}^{N} (\phi_{i}^{(N)})^2 A_{iN}^2, \quad (27c) \]

\[ R[0] = \sum_{i=1}^{N} (\phi_{i}^{(N-1)} \phi_{i}^{(N+1)}) A_{iN}^2, \quad (27d) \]

then we have

\[ P[m] = -\frac{1}{m+1} \left( R[m+1] + Q[m+1] \right), \quad (0 \leq m \leq N - 3), \quad (28a) \]

\[ Q[m] = -\frac{1}{m+1} \left( S[m+1] + P[m+1] \right), \quad (0 \leq m \leq N - 3), \quad (28b) \]

\[ P[N-2] = -\frac{1}{N-1} R[N-1], \quad (28c) \]

\[ Q[N-2] = -\frac{1}{N-1} S[N-1]. \quad (28d) \]

Besides, by successively using the above formulae we can have

\[ P[0] = -\sum_{i=1}^{\lfloor N/2 \rfloor} \frac{1}{(2i-1)!} R[2i-1] + \sum_{i=1}^{\lfloor (N-1)/2 \rfloor} \frac{1}{(2i)!} S[2i], \]

\[ P[1] = -\sum_{i=1}^{\lfloor (N-1)/2 \rfloor} \frac{1}{(2i)!} R[2i] + \sum_{i=2}^{\lfloor N/2 \rfloor} \frac{1}{(2i-1)!} S[2i-1]. \quad (29) \]

The proof for the lemma will be given in Appendix.

**Lemma 6** Suppose that \( A_{jN} \) is the cofactor of \( f = |N-1| \) with respect to the \( j \)-th row and \( N \)-th column,

\[ S[0] + P[0] = \sum_{j=1}^{N} \left( k_{j}^{2(N-1)} \phi_{j,x}^{2} A_{jN}^2 - \sum_{j=1}^{N} \left( \sum_{i_1=1}^{N} \frac{1}{(2i_1)!} k_{j}^{2(N-2)} \phi_{j,x}^{2} A_{jN}^2 \right) \right), \]

we further have

\[ S[0] + P[0] = \sum_{j=1}^{N} k_{j}^{2(N-1)} \phi_{j,x}^{2} A_{jN}^2 - \sum_{j=1}^{N} \sum_{i_1=1}^{N} \frac{1}{(2i_1)!} k_{j}^{2(N-2)} \phi_{j,x}^{2} A_{jN}^2, \]

and for any positive even number \( N \),

\[ S[0] + P[0] = \sum_{j=1}^{N} \left( k_{j}^{2(N-1)} \phi_{j,x}^{2} A_{jN}^2 \right), \quad (32) \]

**Proof** We only prove Eq. (30). The other three formulæ can be derived in a similar way. When \( N \) is positive and odd, from Lemma 5 we have

\[ S[0] + P[0] = S[0] - \sum_{i=1}^{\lfloor N/2 \rfloor} \frac{1}{(2i-1)!} R[2i-1] \]

\[ + \sum_{i=1}^{\lfloor (N-1)/2 \rfloor} \frac{1}{(2i)!} S[2i], \]

\[ = S[0] - R[1] + \frac{1}{2!} S[2] - \frac{1}{3!} R(3) + \cdots \]

\[ + (-1)^{N-2} \frac{1}{(N-2)!} R(N-2) + (-1)^{N-1} \frac{1}{(N-1)!} S(N-1). \]

Then, noting that Eq. (17a) implies

\[ \phi_{j}^{(2s+1)} = k_{j}^{2s} \phi_{j,x}, \quad (s = 0, 1, \ldots) \],  

we further have

\[ \phi_{j}^{(2s+1)} = k_{j}^{2s} \phi_{j,x}, \quad (s = 0, 1, \ldots) \]
\[+ (-1)^{N-1} \sum_{j=1}^{N-1} \left( \sum_{i_1, i_2, \ldots, i_{N-1} = 1}^{N-1} k_{i_1}^2 k_{i_2}^2 \cdots k_{i_{N-1}}^2 \right) (\phi_{j,x})^2 A_j^2 N\]

\[= \sum_{j=1}^{N} \sum_{m=0}^{N-1} (-1)^m k_j^{2(N-m-1)} \left( \sum_{i_1, i_2, \ldots, i_m = 1}^{N-1} k_{i_1}^2 k_{i_2}^2 \cdots k_{i_m}^2 \right) (\phi_{j,x})^2 A_j^2 N = \sum_{j=1}^{N} \left[ \prod_{i=1}^{N} (k_j^2 - k_i^2) \right] \phi_{j,x}^2 A_j^2 N.\]

Thus we get Eq. (30).

**Lemma 7** Suppose that \(A_{jN}\) is the cofactor of \(f = |N - 1|\) with respect to the \(j\)-th row and \(N\)-th column, the entries \(\{\phi_j\}\) satisfy Eq. (17a), and \(P[m], Q[m], R[m], S[m]\) are defined as in Lemma 5. Then, for any positive integer \(N\), we have

\[|N - 2, N|^2 = S[0] + P[0], \quad (35)\]

\[2|N - 2, N + 1||N - 1| - \left( \sum_{i=1}^{N} k_i^2 \right)|N - 1|^2 = R[0] - S(1) - P(1). \quad (36)\]

**Proof** First, for arbitrary positive integer \(N\), we expand \(|N - 2, N|\) by its \(N\)-th column and have

\[|N - 2, N|^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} \phi_j^{(N)} \phi_i^{(N)} A_{jN} A_{iN} = \sum_{j=1}^{N} \left( \sum_{i=j}^{N} + \sum_{i \neq j} \right) \phi_j^{(N)} \phi_i^{(N)} A_{jN} A_{iN}\]

\[= \sum_{j=1}^{N} (\phi_j^{(N)})^2 A_j^2 N + \sum_{i,j=1}^{N} \phi_i^{(N)} \phi_j^{(N)} A_{iN} A_{jN} = S[0] + P[0]. \quad (37)\]

Next, expanding \(|N - 2, N + 1|\) and \(|N - 1|\) by the last column, we have

\[2|N - 2, N + 1||N - 1| = 2 \sum_{i=1}^{N} \phi_i^{(N+1)} \phi_j^{(N-1)} A_{iN} A_{jN} = 2 \sum_{i,j=1}^{N} \phi_i^{(N+1)} \phi_j^{(N-1)} A_{iN} A_{jN}\]

\[= 2 \sum_{i=1}^{N} \phi_i^{(N+1)} \phi_i^{(N-1)} A_{iN}^2 + 2 \sum_{i,j=1}^{N} \phi_i^{(N+1)} \phi_j^{(N-1)} A_{iN} A_{jN} = 2R[0] + 2 \sum_{i,j=1}^{N} \phi_i^{(N+1)} \phi_j^{(N-1)} A_{iN} A_{jN}. \quad (37)\]

Meanwhile,

\[\left( \sum_{i=1}^{N} k_i^2 \right)|N - 1|^2 = \sum_{i=1}^{N} \left( \sum_{i=j}^{N} + \sum_{i \neq j} \right) k_i^2 \phi_i^{(N-1)} A_{iN} A_{jN}\]

\[= \sum_{i=1}^{N} k_i^2 (\phi_i^{(N-1)})^2 A_{iN}^2 + \sum_{i,j=1}^{N} k_i^2 \phi_i^{(N-1)} A_{iN} A_{jN}\]

\[+ \sum_{i,j=1}^{N} k_i^2 \phi_i^{(N-1)} A_{iN} A_{jN} = R[0] + S[1] + 2 \sum_{i,j=1}^{N} k_i^2 \phi_i^{(N-1)} A_{iN} A_{jN} + P[1], \quad (38)\]

where for \(R[0]\) we have made use of \(\phi_j^{(N+1)} = k_j^2 \phi_j^{(N-1)}\). Then, combining Eqs. (37) and (38) yields Eq. (36) and we complete the proof.

Now we can turn to the proof for Eq. (6a). Based on Lemma 2, 4, 6, and 7, we can easily find
\[ f_{xx} f - f_x^2 = \begin{cases} \sum_{j=1}^{N} C_j \left( \prod_{i=1}^{N} (k_j^2 - k_i^2) \right) A_{jN}^2, & (N \text{ is an even number}), \\ \ - \sum_{j=1}^{N} C_j \left( \prod_{i=1}^{N} (k_j^2 - k_i^2) \right) A_{jN}^2, & (N \text{ is an odd number}). \end{cases} \]  

Due to Eq. (6a) we have to choose suitable \( C_j \) so that
\[
\left\{ C_j \prod_{i=1}^{N} (k_j^2 - k_i^2) \right\}_{j=1}^{N}
\]
are always positive when \( N \) is even and negative when \( N \) is odd. To do that, we first set \( 0 < k_1 < k_2 < \cdots < k_N \) without loss of generality and this suggests
\[
\text{sign} \left( \prod_{i=1}^{N} (k_j^2 - k_i^2) \right) = (-1)^{N-j}.
\]

Then, noting that \( C_j = 4k_j^2a_jb_j \) is determined by \( \phi_j \) defined by Eq. (43), we can take \( a_j \) and \( b_j \) such that
\[
\text{sign}[C_j] = (-1)^{N-j}, \quad \text{when } N \text{ is even and sign}[C_j] = (-1)^{N-j-1}, \quad \text{when } N \text{ is odd.}
\]

Particularly, we can take
\[
\phi_j = |a_j| e^{i\xi_j} + (-1)^{N-j} |b_j| e^{-i\xi_j},
\]

as Wronskians entries and when \( |a_j| = |b_j| \neq 0 \) get classical \( N \)-soliton solutions.\(^{[15]} \) Thus, we complete the proof for Theorem 1.

In addition, we note that by the same procedure given in Ref. [16], we can rewrite \( f \) and \( g \) given by Eq. (16) into the form of exponential polynomials given by Eq. (13), and so we get classical solitons. If we do not care of whether \( \varphi \) is real or complex, (but still keep \( u \) real), we can take
\[
\phi_j = a_j e^{i\theta_j} + b_j e^{-i\theta_j},
\]

where
\[
\theta_j = k_j x - 4k_j^2 t + \theta_j^{(0)},
\]

with arbitrary real constants \( k_j \) and \( \theta_j^{(0)} \) while \( b_j = a_j^* \). Here, * stands for complex conjugate. The corresponding solutions are referred to as positons.\(^{[17]} \) When \( N \) is even, we can also take, for \( j = 1, 2, \ldots, N/2, \)
\[
\phi_{2j-1} = a_j e^{i\xi_j} + b_j e^{-i\xi_j}, \quad \phi_{2j} = \phi_{2j-1}^*, \quad \xi_j = k_j x + 4k_j^2 t + \xi_j^{(0)},
\]

where \( a_j, b_j, k_j, \) and \( \xi_j^{(0)} \) are arbitrary complex constants but specially asking \( \text{Re}[k_j] \cdot \text{Im}[k_j] \neq 0 \). In this case, we get complexitons or breathers\(^{[18-19]} \) of the KdV equation.

### 4 Conclusions and Remarks

We have shown that the Lax pair of the KdV equation together with the Bargmann constraint can be solved by

Hirota method and Wronskian technique. What is interesting and meaningful behind this is that the potential \( u \) in the stationary Schrödinger equation (2a) are expressed as a summation of squares of wave function \( \phi_j \) and these wave functions are nothing but Wronskians in lower order.

The letter reviewed the Bargmann constraint for the KdV equation from bilinear point of view. Besides from the GLM equation, the constraint can also be derived from Lax pair or bilinear Bäcklund transformation. This fact is reasonable since the GLM equation and bilinear Bäcklund transformation are connected with Lax pair, and the GLM equation is also solvable by bilinear approach.\(^{[20]} \) We hope our discussions will be helpful for getting more insight into integrable systems.

In the paper, we have developed a procedure to achieve the Wronskian verification for the bilinear Bargmann constraint equation (6a). This procedure is general and can be shared by those equations which have the same bi-linear Bargmann constraint and the Wronskian entry \( \phi_j \) satisfying Eq. (17a), i.e., \( \phi_{xx} = k_j^2 \phi_j \). Such equations are the KP equation, the 5th order KdV equation, and so on. Since for many soliton systems their Lax pairs and bilinear Bäcklund transformations are derived from each other,\(^{[12]} \) we can immediately conclude that the Bargmann-Lax systems of such equations are also solvable in the same bilinear approach. For the mKdV equation and the sine-Gordon which have bilinear constraint equations different from Eq. (6a), our procedure given in the letter will provide many hints includes for achieving verifications. Besides, we can also try other constraints, such as Neumann constraint and symmetry constraint. These will be investigated elsewhere.

### Appendix

Here we give the proof for Lemma 5.

Noting that
\[
\sum_{j=1}^{N} \phi_j^{(N-s)} A_{jN} = \begin{cases} 0, & s = 1, \\ \lceil N-1 \rceil, & s = 2, 3, \ldots, N, \end{cases}
\]
we have
\[
\phi_j^{(N-s)} A_{jN} = - \sum_{i=1}^{N} \phi_i^{(N-s)} A_{iN}, \quad (s = 2, 3, \ldots, N). \quad (A2)
\]

For \( P[0] \) defined by Eq. (27a), by virtue of \( \phi_j^{(N)} = \)
Further, by means of Eq. (A2),

\[
P(0) = - \sum_{i,j=1}^{N} k^{2(N)}_{i,j} A_{iN} A_{jN} 
\]

\[
= -\sum_{i=1}^{N} \left( \sum_{j\neq i} + \sum_{j\neq i} \right) k^{2(N)}_{j} \phi^{(N-2)}_{i} A_{jN} A_{iN} 
\]

\[
= -\sum_{i,j=1}^{N} k^{2(N-2)}_{j} \phi^{(N-1)}_{i} A_{iN} A_{jN} 
\]

\[
= -\sum_{i,j,l=1}^{N} k^{2(N-2)}_{i,2} \phi^{(N)}_{i} A_{i2} A_{i3} A_{jN} 
\]

which further yields

\[
Q[1] = - \frac{1}{2} (S[2] + P[2]) ,
\]

i.e., formula (28b) for \( m = 1 \).

As examples by \( P[0] \) and \( Q[1] \) we give a detailed proof. In fact, those formulae in Eq. (28) can be derived in a quite similar way and here we skip the proof.

References