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Rational and Periodic Solutions for a (2+1)-Dimensional Breaking Soliton Equation Associated with ZS-AKNS Hierarchy*

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Abstract The double Wronskian solutions whose entries satisfy matrix equation for a (2+1)-dimensional breaking soliton equation ((2+1)DBSE) associated with the ZS-AKNS hierarchy are derived through the Wronskian technique. Rational and periodic solutions for (2+1)DBSE are obtained by taking special cases in general double Wronskian solutions.

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1 Introduction

The breaking soliton equations (BSEs) are a kind of nonlinear evolution equations which can be used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long-wave propagating along the x-axis.[1] One of the most famous BSEs is described as[2]

\[ u_{xt} - 4u_xu_{xy} - 2u_yu_{xx} + u_{xxy} = 0. \]  (1)

The algebraic properties and Lax integrability, the existence of localized coherent structures, solutions, soliton-like solutions, the Hamiltonian structure, and the Painlevé property have been discussed in Refs. [3–7]. Reference [3] has shown that the self-dual Yang–Mills equation belongs to the class of BSEs.

The (2+1)-dimensional breaking soliton equation (2+1)DBSE) associated with the ZS-AKNS hierarchy,[8–9] which was proposed by Bogoyavlenskii,[1] takes the form

\[ q_t = -q_{xy} + 2q\partial_x^{-1}(qr)_y, \]
\[ r_t = q_{xy} - 2r\partial_x^{-1}(qr)_y, \]  (2)

where \( q \) and \( r \) are potential functions, \( \partial = \partial / \partial x, \partial \partial^{-1} = \partial^{-1} \partial = 1 \). In 1993, Li et al. constructed many symmetries by infinitesimal dressing method,[10] recently, the multisoliton solutions were obtained by Hirota’s bilinear method.[11] In this letter, we turn our interest to the understanding of the properties belong to its solutions. We would like to argue that such a discussion is more than worthwhile as it allows one to uncover a lot of peculiar phenomena.

Our discussion bases on the Wronski determinant (Wronskian), which was first introduced to describing solitary waves by Freeman and Nimmo.[12–13] Recently Chen et al. extended the traditional condition equation to the arbitrary matrix equation and they obtained the rational and complexon solutions[14] in terms of double Wronskian form for AKNS system.[15] In this paper, by making use of Chen’s method, we discuss the rational solutions and periodic solutions in terms of double Wronskian for the (2+1)DBSE.

The paper is organized as follows. In Sec. 2, the hierarchy of (2+1)DBSEs is derived and the Lax pair of (2) is given. In Sec. 3, we give the double Wronskian solution of (2) whose entries satisfy general matrix equation. The rational solutions and periodic solutions are obtained by taking special cases in double Wronskian solutions. A conclusion is given in the final section.

2 Lax Integrability of BSE

In this section, we derive the hierarchy of the (2+1)-dimensional breaking soliton equations which concerns with the ZS-AKNS soliton hierarchy closely. Consider the ZS-AKNS spectral problem[9]

\[ \psi_x = M\psi, \quad M = \begin{pmatrix} \lambda & q \\ r & \lambda \end{pmatrix}, \]  (3)

where \( q \) and \( r \) are potential functions, the spectral parameter \( \lambda = \lambda(y,t) \) is independent of \( x \). The adjoint time evolution

\[ \psi_t = 2\lambda \psi_y + N\psi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \]  (4)

From (3) and (4), we know there have two main characteristic features of BSEs, one is the spectral parameter \( \lambda \)
possesses so-called breaking behavior, in other word the spectral value becomes a multivalued function about \( y \) and \( t \). The other feature is the time evolution not only depends on \( N \) but also \( \psi_y \). Consequently, the solutions of these equations may be multivalued too and have much larger types of properties than AKNS hierarchy.

The compatibility of (3) and (4), i.e. the zero-curvature equation is
\[
M_t - N_x + [M, N] - 2\lambda M_y = 0. \tag{5}
\]
From (5), we have
\[
\begin{align*}
\left( \frac{q}{r} \right)_t &= L \left( -\frac{B}{C} \right) - 2\lambda \left( -\frac{B}{C} \right) \\
&\quad + 2(2\lambda\psi_y - \lambda_t)\sigma \left( \frac{xq}{xr} \right) \\
&\quad + 2\lambda \left( \frac{q}{r} \right)_y - 2A_0\sigma \left( \frac{q}{r} \right), \tag{6}
\end{align*}
\]
where \( A_0 \) is a constant and
\[
L = \sigma \partial + 2 \left( \frac{q}{-r} \right) \partial_{-1}(r, q), \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}
\]
For the purpose of finding soliton hierarchy, we expand \((B, C)^T\) as
\[
\left( \frac{-B}{C} \right) = \sum_{j=1}^{n} \left( \frac{-b_j}{c_j} \right) \lambda^{n-j}. \tag{8}
\]
To get the isospectral hierarchy of (2+1)DBSE, we take
\[
2\lambda\psi_y - \lambda_t = 0 \quad \text{which is Riemann equation and } A_0 = 0.
\]

\[
N\big|_{(q,r) = (0,0)} = \begin{pmatrix} -(1/2)(\gamma(2\lambda) + \omega(2\lambda)x) \\ 0 \end{pmatrix}, \tag{14}
\]
where \( L \) is the recursive operator (7).

3 Rational and Periodic Solutions for (2+1)DBSE

In this section, we investigate two kinds of solutions for (2) rational solution and periodic solution. First of all, we give the double Wronskian solution for (2+1)DBSE. Base on it, we discuss the rational and periodic solutions.

Equation (2) holds the bilinear form
\[
(D_t + D_x D_y)g \cdot f = 0, \quad (D_t - D_x D_y)h \cdot f = 0,
\]
\[
D_x^2 f \cdot f = 2gh, \tag{15}
\]
by introducing the dependent variable transformation
\[
q = \frac{g}{f}, \quad r = -\frac{h}{f}, \tag{16}
\]
where \( D \) is the Hirota’s bilinear operator.\(^{[16]}\)

The double Wronskian solution for (2+1)DBSE can be described as the following theorem

**Theorem 1** Let \( \gamma(\lambda) \) and \( \omega(\lambda) \) are \( n \)-th degree Polynomial independent of \( x \), if \( \lambda \) satisfies Riemann equation
\[
\lambda(y) - \lambda_t = \frac{1}{2}\omega(2\lambda), \tag{12}
\]
then the hierarchy of (2+1)DBSE associated with ZS-AKNS problem is
\[
\frac{q}{r}_t = \gamma(\lambda) \left( \frac{-q}{r} \right) + \omega(\lambda) \left( \frac{-xq}{xr} \right) + L \left( \frac{-q}{r} \right)_y, \tag{13}
\]
with the boundary condition
\[
\begin{pmatrix} 0 \\ (1/2)(\gamma(2\lambda) + \omega(2\lambda)x) \end{pmatrix},
\]
respectively, where \( \Gamma \) is an arbitrary constant matrix whose rank is \((N + M + 2)\). In this paper, we call (18) Wronskian condition equations of (15).

The proof is given in Ref. [11]. Besides, in theorem 1, we have adopted the compact notation of Wronskian as Freeman and Nimmo did\(^{[12-13]}\).

3.1 Rational Solutions for (2+1)DBSE

The Wronskian condition equations (18) have general solutions
\( \varphi = e^{-\Gamma x + \Gamma^2 y + 2\Gamma^3 t} C, \quad \psi = e^{\Gamma x - \Gamma^3 y - 2\Gamma^4 t} D, \) \hspace{1cm} (20)

where \( C = (C_1, C_2, \ldots, C_{N+M+2})^T, \quad D = (D_1, D_2, \ldots, D_{N+M+2})^T \) are constant vectors. Equation (20) admits the Taylor expansions

\[
\varphi = \sum_{s=0}^{N+M+2} \sum_{l=0}^s \sum_{n=0}^l (-1)^{s-3l-2n} \frac{2^l x^s - 3l - 2n y^n t^l}{n!!(s - 3l - 2n)!} \Gamma^s C,
\]

\[
\psi = \sum_{s=0}^{N+M+2} \sum_{l=0}^s \sum_{n=0}^l (-1)^{l+n} \frac{2^l x^s - 3l - 2n y^n t^l}{n!!(s - 3l - 2n)!} \Gamma^s D. \tag{21}
\]

### 3.1.1 Soliton Solutions for (2+1)DBSE

Suppose the \( \Gamma \) is diagonal matrix, i.e. \( \Gamma = \text{diag}(k_1, k_2, \ldots, k_{N+M+2}) \), obviously we have

\[
\varphi_j = C_j e^{-k_j x + k_j^2 y + 2k_j^3 t}, \quad \psi_j = D_j e^{k_j x - k_j^3 y - 2k_j^4 t}
\] \hspace{1cm} (22)

\( j = 1, 2, \ldots, N + M + 2 \).

The Wronskians composed by (22) are the normal soliton solutions for (2).

### 3.1.2 Rational Solutions Related to \( \Gamma \)

If we consider \( \Gamma \) as

\[
\Gamma = \begin{pmatrix}
0 & 0 & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 1 & 0 & \cdots & \cdots \\
0 & 0 & 0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 1 & 0 \\
& \cdots & \ddots & \ddots & \ddots & \ddots \\
0 & & & & \vdots & \ddots \\
0 & & & & & \ddots \\
& & & & & & \ddots \\
\end{pmatrix}_{(N+M+2) \times (N+M+2)}.
\tag{23}
\]

Noticing that \( \Gamma^{N+M+2} = 0 \), the series (21) are truncated as

\[
\varphi = \sum_{s=0}^{N+M+2} \sum_{l=0}^s \sum_{n=0}^l (-1)^{s-3l-2n} \frac{2^l x^s - 3l - 2n y^n t^l}{n!!(s - 3l - 2n)!} \Gamma^s C,
\]

\[
\psi = \sum_{s=0}^{N+M+2} \sum_{l=0}^s \sum_{n=0}^l (-1)^{l+n} \frac{2^l x^s - 3l - 2n y^n t^l}{n!!(s - 3l - 2n)!} \Gamma^s D. \tag{24}
\]

As a special result, let \( C_1 = 1, \quad C_j = 0, \quad j = 2, 3, \ldots, N + M + 2 \), then Eqs. (24) reduce to

\[
\varphi_j = \sum_{l=0}^{j-1} \sum_{n=0}^{j-3l-2n-1} (-1)^{j-3l-2n-1} \frac{2^l x^j - 3l - 2n - 1 y^n t^l}{n!!(j - 3l - 2n - 1)!},
\]

\[
\psi_j = \sum_{l=0}^{j-1} \sum_{n=0}^{j-3l-2n-1} (-1)^{l+n} \frac{2^l x^j - 3l - 2n - 1 y^n t^l}{n!!(j - 3l - 2n - 1)!}. \tag{25}
\]

The double Wronskians composed by (25) will generate the rational solutions for (2) through the transformation (16). For examples, when \( N + M = 0; \quad N = 1, M = 0; \quad N = 0, M = 1; \quad N = M = 1; \quad N = 2, M = 0; \quad N = 0, M = 2 \), we obtain the rational solution for (2), respectively

\[
q = r = -\frac{1}{x}, \tag{26a}
\]

\[
q = \frac{1}{x^2 - y}, \quad r = \frac{x^2 + y}{x^2 - y}. \tag{26b}
\]

![Fig. 1](image1) The shape and motion of \( q \) (a) and \( r \) (b) in (26c), when \( t = 1 \).

![Fig. 2](image2) The shape and motion of \( q \) (on the left) in (26d) and \( r \) (on the right) in (26f), when \( t = 1 \).
3.2 Periodic Solutions for \((2+1)\)DBSE

In 1996, Porubov proposed Weierstrass elliptic function expansion method, Liu et al. proposed Jacobi elliptic function expansion methods obtained some exact periodic solutions of some nonlinear evolution equations\cite{17-18} different from them, we deduce the periodic solution from the view of double Wronskian solution.

If \(\Gamma\) has the form as
\[
\Gamma = \begin{pmatrix}
\alpha & -\beta \\
\beta & \alpha \\
\end{pmatrix} = \alpha I_2 + \beta \sigma_2,
\]
\[
I_2 = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
\end{pmatrix},
\]
by a direct inserting into (21), we have
\[
\varphi = e^{-\alpha x + (\alpha^2 - \beta^2) y + 2(3\alpha^2 - 5\beta^2) t} \times (\cos[\beta x + 2\alpha \beta y + 2(3\alpha^2 \beta - \beta^3) t] \sigma_2) D.
\]

Furthermore, consider \(\Gamma\) is Jordan block matrix
\[
\Gamma = \begin{pmatrix}
J_1 & 0 \\
0 & J_1 \\
\end{pmatrix},
\]
and
\[
J_k = \begin{pmatrix}
\Gamma_k & 0 \\
0 & \Gamma_k \\
\end{pmatrix}, \quad \Gamma_k = \begin{pmatrix}
\alpha_k & -\beta_k \\
\beta_k & \alpha_k \\
\end{pmatrix}.
\]

Then \(J_k\) could be expressed as
\[
J_k = \begin{pmatrix}
I_2 & 0 \\
I_2 \partial_{\alpha_k} & I_2 \\
\vdots & \vdots \\
\frac{1}{(k-1)!} I_2 \partial_{\alpha_k}^{k-1} & \frac{1}{k!} I_2 \partial_{\alpha_k}^k & I_2 \\
\end{pmatrix}, \quad \Gamma^s = T_{\alpha_k} \Gamma^s.
\]

where \(\Gamma' = \text{diag}(\Gamma_k, \Gamma_k, \ldots, \Gamma_k)_{2k}\). Replace \(\Gamma^s\) in (21) with (31), by a direct but complicated calculation, we obtain the following expressions of \(\varphi\) and \(\psi\)
\[
\varphi_j(\alpha_k) = \sum_{s=1}^{j} \frac{1}{(j-s)!} \partial_{\alpha_k}^{j-s} e^{\xi_k} \left( c_{s1} \cos \theta_k - c_{s2} \sin \theta_k, c_{s1} \sin \theta_k + c_{s2} \cos \theta_k \right),
\]
\[
\psi_j(\beta_k) = \sum_{s=1}^{j} \frac{1}{(j-s)!} \partial_{\beta_k}^{j-s} e^{\xi_k} \left( c_{s1} \cos \theta_k + c_{s2} \sin \theta_k, c_{s1} \sin \theta_k + c_{s2} \cos \theta_k \right),
\]
where \(\xi_k = -\alpha_k x + (\alpha_k^2 - \beta_k^2) y + 2(\alpha_k^2 - 2\alpha_k \beta_k t + 2(3\alpha_k \beta - \beta_k) t)\) and \(\theta_k = -\beta_k x + 2\alpha_k \beta_k y + 2(3\alpha_k \beta - \beta_k) t\).

Similarly, we can find
\[
\varphi_j(\beta_k) = \sum_{s=1}^{j} \frac{1}{(j-s)!} \partial_{\beta_k}^{j-s} e^{\xi_k} \left( c_{s1} \cos \theta_k - c_{s2} \sin \theta_k, c_{s1} \sin \theta_k + c_{s2} \cos \theta_k \right),
\]
\[
\psi_j(\beta_k) = \sum_{s=1}^{j} \frac{1}{(j-s)!} \partial_{\beta_k}^{j-s} e^{\xi_k} \left( c_{s1} \cos \theta_k + c_{s2} \sin \theta_k, c_{s1} \sin \theta_k + c_{s2} \cos \theta_k \right).
\]

It is easy to show that (32) and (33) are equivalent, so we get the complexiton solutions\cite{14,19} for (2) where the Wronskian entries satisfy
\[
\varphi = (\varphi_1^T(\alpha_1), \ldots, \varphi_{l_1}^T(\alpha_1); \varphi_1^T(\alpha_2), \ldots, \varphi_{l_2}^T(\alpha_2); \ldots; \varphi_1^T(\alpha_h), \ldots, \varphi_{l_h}^T(\alpha_h))^T,
\]
\[
\psi = (\psi_1^T(\alpha_1), \ldots, \psi_{l_1}^T(\alpha_1); \psi_1^T(\alpha_2), \ldots, \psi_{l_2}^T(\alpha_2); \ldots; \psi_1^T(\alpha_h), \ldots, \psi_{l_h}^T(\alpha_h))^T,
\]
where \(l_1 + l_2 + \cdots + l_h = N + M + 2\). Specially, if \(N = M = 0, \ l_1 = 1, \ l_j = 0, \ j = 2, 3, \ldots, h\), we can obtain the periodic solution
\[
\varphi = e^{\xi_1} \left( c_{11} \cos \theta_1 - c_{12} \sin \theta_1, c_{11} \sin \theta_1 + c_{12} \cos \theta_1 \right), \quad \psi = e^{-\xi_1} \left( c_{11} \cos \theta_1 + c_{12} \sin \theta_1, -c_{11} \sin \theta_1 + c_{12} \cos \theta_1 \right).
\]

Hence
\[
f = |\varphi; \psi| = -(c_{11}^2 + c_{12}^2) \sin 2\theta_1, \quad g = 2|\varphi; \partial_x \varphi| = -2\beta_1 (c_{11}^2 + c_{12}^2) e^{2\xi_1},
\]
\[
h = 2|\varphi; \partial_x \varphi| = 2\beta_1 (c_{11}^2 + c_{12}^2) e^{-2\xi_1},
\]

Noticing the transformation (16), we get the periodic solution as follows
\[
q = 2\beta_1 e^{2\xi_1} \csc 2\theta_1, \quad r = 2\beta_1 e^{-2\xi_1} \csc 2\theta_1.
\]
We describe the periodic solution in Figs. 3 and 4.

**Fig. 3** The shape and motion of $q$ (a) and $r$ (b) in (36), when $t = 1$.

**Fig. 4** The shape and motion of $q$ (a) and $r$ (b) in (36), when $x = -0.5$, $t = 0$.

4 Conclusion

In summary, we have given the hierarchy of (2+1)DBSEs. The double Wronskian solutions for (2) which satisfy condition equations are obtained. Based on the solution, we deduce the rational solutions and the periodic solutions through special choosing of the condition matrix $\Gamma$.

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