Casoratian Solutions to Toda Lattice Via Its Bäcklund Transformation*

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Abstract Generalized Casoratian condition and Casoratian solutions of the Toda lattice are given in terms of its bilinear Bäcklund transformation. By choosing suitable Casoratian entries and parameter in the bilinear Bäcklund transformation, we can give transformations among many kinds of solutions.

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1 Introduction

Bäcklund Transformation (BT) is a useful way to generate new solutions from the old one for nonlinear evolution equations. In 1883, Bäcklund first found that the sine-Gordon equation \( u_{xt} = \sin u \) keeps invariant under the transformation

\[
\begin{align*}
    u'_x &= -u_x + 2a \sin \left( \frac{u' - u}{2} \right), \\
    u'_t &= u_t + \frac{2}{a} \sin \left( \frac{u' + u}{2} \right).
\end{align*}
\]  

(1)

Now Eq. (1) is known as the Bäcklund transformation of the sine-Gordon equation, which also provides a relation for the solutions \( u \) and \( u' \). In 1971, based on Eq. (1) Lamb\(^4\) gave a nonlinear superposition formula for the solutions of the sine-Gordon equation. Then, in 1973 Wahlquist and Estabrook\(^2\) derived a BT for the KdV equation and also gave a nonlinear superposition formula. Soon after, Chen\(^3\), Hirota\(^4\), Wadati\(^5\) et al. obtained BTs for more soliton equations and discussed the relationship between equations, Lax pairs, and BTs. More introductions can be found from Refs. [6–7]. In general, the BT is an equation set with lower order than the original equation(s). Difficulty usually arise when we again and again use BT to get new solutions since the equation set we need to solve will get more and more complicated. One way to conquer such difficulties is to work out nonlinear superposition formulas\(^2–3\) between several solutions. On the other hand, Hirota provided a bilinear form\(^4\) for BTs. Such a BT admits explicit \( N \)-solitons in the form of exponential polynomials\(^8\) or can provide transformations between \((N-1)\)-solitons and \(N\)-solitons in Wronskian (or Casoratian for discrete case) form\(^9–10\).

Besides solitons, Wronskian can provide many kinds of solutions, such as positons, negatons, rational solutions and complexitons, (See Refs. [11–13]). In this paper we investigate solutions of the Toda lattice in Casoratian form via its bilinear BT. Our work will mainly focus on what the generalized Casoratian condition is under the bilinear BT and how to choose parameters for Casoratian condition.

The paper is organized as follows. Section 2 investigates properties of a kind of matrix with special form. In Sec. 3 we discuss Casoratian solutions of Toda lattice via bilinear BT. We give Casoratian entries and transformations among many kinds of solutions in Sec. 4. Finally, conclusions are given in Sec. 5.

2 A Kind of Matrix with Special Form and Their Properties

First, we will give a kind of matrix \( \mathcal{G}_s \), which had been defined in Ref. [14] as the following form:

\[
\mathcal{G}_s = \left\{ A_{s \times s}| A_{s \times s} = \begin{pmatrix} \hat{A} & 0 \\ \alpha & a_{ss} \end{pmatrix} \right\},
\]  

(2)

where \( \hat{A} \) is an \((s-1)(s-1)\) matrix with arbitrary complex numbers as entries, the complex vector \( \alpha = (a_1, \ldots, a_{s-1}) \), \( 0 \) stands for an \( s-1 \) order column zero vector, and \( a_{ss} \) is a complex scalar.

Many properties of \( \mathcal{G}_s \) will play important roles in the following sections. We list them with some propositions, where we skip over their proofs and the detailed discussions can be found from Ref. [14].

Proposition 1 \( \mathcal{G}_s \) defined by Eq. (2) forms a semigroup with identity with respect to matrix multiplication and inverse.

Proposition 2 For any given \( A \in \mathcal{G}_s \), there exists a transformation matrix \( B \in \mathcal{G}_s \) such that \( A = B^{-1}CB \), where \( C \) is an \( s \times s \) lower triangular matrix.

Proposition 3 For any given \( A \in \mathcal{G}_N \), there exists a matrix \( T \in \mathcal{G}_N \) satisfying \( T^{-1}AT = \Gamma \), where \( \Gamma \) is defined

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as the following form
\[ \Gamma = \begin{pmatrix} \Gamma_{N-1} & 0 \\ \gamma & a_{NN} \end{pmatrix}, \tag{3} \]
and \( \Gamma_{N-1} \) is the canonical form of \( \hat{A} \). Here we call Eq. (3) quasi-canonical form of \( A \).

Now we expand \( \mathcal{G}_s \) to a function semi-group \( \mathcal{G}_s[t] \) which consists \( A_{s \times s} \) defined as Eq. (2) but each non-zero entry of \( A_{s \times s} \) is a functions of \( t \) instead. For the function matrix in \( \mathcal{G}_s[t] \), we have the following result.

**Proposition 4** Suppose that \( B(t) = (b_{ij}(t)) \in \mathcal{G}_N[t] \) with \( b_{NN}(t) = 0 \) and each \( b_{ij}(t) \in [a, b] \) where \( a \) and \( b \) can be infinite. Then, there exists a non-singular \( N \times N \) \( t \)-dependent matrix \( H(t) = (h_{ij}(t)) \in \mathcal{G}_N[t] \), satisfying
\[ H(t)_t = -H(t)B(t). \tag{4} \]

### 3 Casoratian Solutions to Toda Lattice Via BT

#### 3.1 Bilinear BT of Toda Lattice

The Toda lattice\(^{[17]} \) is
\[ \frac{\partial^2}{\partial t^2} \ln[1 + V(t)] = V_{n+1}(t) + V_{n-1}(t) - 2V_n(t), \tag{5} \]
with its bilinear form\(^{[18]} \)
\[ \left[D_n^2 - 4\sinh^2 \left(\frac{1}{2}D_n\right)\right]f_n \cdot f_n = 0, \tag{6} \]
under the transformation\(^{[12]} \)
\[ V_n = \frac{f_{n+1}f_{n-1}}{f_n^2} - 1. \tag{7} \]

The BT between two solutions \( g \) and \( f \) of Eq. (6) is defined by the pair of relations:\(^{[10]} \)
\[ (D_t + e^{-D_n} - 2\cosh \alpha)f_n \cdot g_n = 0, \tag{8a} \]
\[ (D_t - e^{-D_n})f_n \cdot g_{n+1} = 0, \tag{8b} \]
where \( e^{-D_n}f_n \cdot g_n = f_{n+\epsilon}g_{n-\epsilon} \), and \( \alpha \) is a parameter.

#### 3.2 Casoratian Condition w.r.t. BT for Toda Lattice

An \( N \times N \) Casoratian is defined as
\[ \mathrm{Cas}(\phi_1(n, t), \phi_2(n, t), \ldots, \phi_N(n, t)) = \begin{vmatrix} \phi_1(n, t) & \phi_1(n+1, t) & \cdots & \phi_1(n+N-1, t) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N(n, t) & \phi_N(n+1, t) & \cdots & \phi_N(n+N-1, t) \end{vmatrix}. \]

It can be denoted by the following compact form
\[ \mathrm{Cas}(\phi(n, t)) = |\phi(n, t), \phi(n+1, t), \ldots, \phi(n+N-1, t)| = |0, 1, \ldots, N-1| = |N-1|, \]
where \( \phi(n, t) = (\phi_1(n, t), \phi_2(n, t), \ldots, \phi_N(n, t))^T \). Similarly, we also define two \( N \times N \) matrices as following
\[ |\phi(n+1, t), \ldots, \phi(n+N, t)| = |1, \ldots, N| = |N|, \]
\[ |\phi(n, t), \phi(n+1, t), \ldots, \phi(n+N-2, t), \tau_j| = |0, 1, \ldots, N-2, \tau_j| = |N-2, \tau_j|, \]

where
\[ \tau_j = (\delta_{j,1}, \delta_{j,2}, \ldots, \delta_{j,N}), \delta_{j,i} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \]

If \( f_n = |N-1| \) and each entry \( \phi_j \) satisfies\(^{[10]} \)
\[ \phi_j(n+1, t) + \phi_j(n-1, t) = k_j \phi_j(n, t), \]
\[ \pm \phi_j(n, t) = \phi_j(n+1, t), \tag{9} \]
with arbitrary parameter \( k_j \), then such an \( f_n \) solves the bilinear Toda lattice (6).

Under the same Casoratian condition as Eq. (9), the bilinear BT (8) provides a transformation between \( f_n = |N-1| \) and \( g_n = |N-2, \tau_N| \), where we take \( 2\cosh \alpha = k_N \). In this case, \( f_n \) and \( g_n \) provide \( N \)-solitons and \( (N-1) \)-solitons of the Toda lattice (5) through Eq. (7) and
\[ V_n = \frac{g_{n+1}g_{n-1}}{g_n^2} - 1, \tag{10} \]
respectively.

A recent review\(^{[12]} \) shows Casoratian solution of Toda lattice via the bilinear form (6), which can be given as the following Theorem.

**Theorem 1** The following Casoratian \( f_n \) solves the bilinear Toda lattice (6):
\[ f_n = \mathrm{Cas}(\phi(n, t)) \]
\[ = \mathrm{Cas}(\phi_1(n, t), \phi_2(n, t), \ldots, \phi_N(n, t)) = |N-1|, \tag{11} \]
where \( \phi(n, t) \) satisfies
\[ \phi(n+1, t) + \phi(n-1, t) = A(t)\phi(n, t), \]
\[ \pm \phi(n, t) = a\phi(n+1, t) + c\phi(n-1, t) + B(t)\phi(n, t), \tag{12} \]
with \( A(t) = (A_{ij})_{N \times N} \) and \( B(t) = (B_{ij})_{N \times N} \) are two arbitrary \( N \times N \) matrices of \( t \) but independent of \( n, B(t) \) satisfies
\[ (\mathrm{Tr} B(t))_t = 0, \tag{13} \]
where \( \mathrm{Tr} (\cdot) \) means trace and the constant pair \((a, c)\) is equal to \((1/2, -1/2)\) or \((1, 0)\) or \((0, 1)\). Considering that Eq. (12) should be solvable, \( A(t) \) and \( B(t) \) must further satisfy
\[ A(t)_t + [A(t), B(t)] = 0. \tag{14} \]

In what follows we investigate the Casoratian condition on the basis of the bilinear BT (8). Let us first give the following theorem.
**Theorem 2** Casoratian solutions to Eq. (8) are given as

\[ f_n = |N - 1|, \quad g_n = |N - 2|, \quad \tau_N, \]  

(15)

provided that its entries

\[ \phi(n, t) = (\phi_1(n, t), \phi_2(n, t), \ldots, \phi_N(n, t))^T \]

satisfy

\[ \phi(n + 1, t) + \phi(n - 1, t) = A(t)\phi(n, t), \]  

(16a)

\[ \pm \phi(n, t) = a_0\phi(n + 1, t) + c\phi(n - 1, t) + B(t)\phi(n, t), \]  

(16b)

where the constant \( c = a \pm 1, a \) is an arbitrary constant, \( A(t) = (a_{ij})_N \) and \( B(t) = (b_{ij})_N \) are two \( N \times N \) matrices in \( G_N \), depending on \( t \) but independent of \( n \), satisfying

\[ A(t)^N + [A(t), B(t)] = 0, \]  

(17)

and in Eq. (16) we take \( b_{NN} = 0 \), \( a_{NN}(t) = 2 \cosh \alpha \), which is a constant due to Eq. (17).

In the paper, we call Eqs. (16) and (17) the Casoratian condition w.r.t. bilinear BT of the Toda lattice.

To smoothly discuss Casoratian solutions of Eq. (8), we start from the following Lemmas.

**Lemma 1** Suppose \( M \) is an \( N \times (N - 2) \) matrix, \( a, b, c, d \) are \( N \)-order column vectors. Then, we have

\[ |M, a, b| = |M, a, c| = |M, a, d| = |M, b, d| = 0, \]

(18)

**Lemma 2** Suppose that \( \Xi \) is an \( N \times N \) matrix with column vector set \( \{ \xi_j \} \); \( \Omega \) is an \( N \times N \) operator matrix with column vector set \( \{ \Omega_j \} \) and each entry \( \Omega_{j,s} \) is an operator. Then we have

\[ \sum_{j=1}^N |\Omega_j \ast \Xi| = \sum_{j=1}^N (|\Omega_j^T| + \Xi|), \]  

(19)

where for any \( N \)-order column vectors \( A_j \) and \( B_j \) we define

\[ A_j \circ B_j = (A_{1,j}, B_{1,j}, A_{2,j}, B_{2,j}, \ldots, A_{N,j}, B_{N,j})^T, \]

\[ |A_j \ast \Xi| = |\Xi_1, \ldots, \Xi_{j-1}, A_j \circ \Xi_j, \Xi_{j+1}, \ldots, \Xi_N|. \]

By virtue of Lemma 2 we can have several identities. For example, taking \( \Xi = |N - 1| \) and \( \Omega_j = E + E^{-1} \)

\[ |(\text{Tr}(A))|N - 1||N - 2|, \quad \tau_N| = |N - 1||\text{Tr}(A)| - a_{NN}|N - 2|, \quad \tau_N = a_{NN}|N - 2|, \]

under the conditions \( A(t) \in G_N \), \( B(t) \in G_N \), \( b_{NN} = 0 \), and \( 2 \cosh \alpha = a_{NN} \), we have

\[ 2 \cosh \alpha|N - 1||N - 2|, \quad \tau_N = |N - 1||N - 2|, \quad \tau_N = |N - 1||N - 2| + |N - 3|, \quad \tau_N = |N - 1||N - 2| + |N - 1||N - 2|, \]

Then taking \( f_n = |N - 1| \) and \( g_n = |N - 2| \) in Eq. (8a) yields

\[ \text{l.h.s. Eq. (8a)} = |N - 1||N - 2|, \quad \tau_N = |N - 1||N - 2|, \quad \tau_N = |N - 1||N - 2| + |N - 1||N - 2|, \]

(19)

which is zero in the light of Lemma 1. Thus Eq. (8a) is proved.

Similarly, we can verify Eq. (8b).

Thus, combining Casoratian condition (16) and Lemma 1 we complete our proof for Theorem 2.

**3.3 Further Discussion to Casoratian Condition**

In this section, we will simplify the Casoratian condition and try to get a simple version.

First, we give the following Lemma.

**Lemma 4** Suppose that matrices \( g_n(\phi) \) and \( f_n(\phi) \) are
defined as Eq. (15) with entry vector \( \phi \) and
\[
\psi = P(t)\phi ,
\]
where \( P(t) = (P_{ij}(t))_{N \times N} \) is a non-singular matrix in \( \mathcal{G}_N[t] \). Then we have
\[
f_n(\psi) = |P(t)|f_n(\phi) , \quad g_n(\psi) = \frac{|P(t)|}{F_N(t)}g_n(\phi) .
\]  

Thus, by the Lemma 4 and the transformation (7), we conclude that \( f_n(\phi) \) and \( f_n(\psi) \) generate same solutions to the Toda lattice (5). So do \( g_n(\phi) \) and \( g_n(\psi) \).

Next we turn to simplify the Casoratian condition (16).

For \( B(t) \) in Eq. (22), by virtue of Proposition 4, there exists a non-singular \( t \)-dependent matrix \( H(t) \) in \( \mathcal{G}_N[t] \) such that
\[
H(t)_t = -H(t)B(t) .
\]

If setting
\[
\psi = H(t)\phi ,
\]
Eq. (16) can be written as
\[
\psi(n + 1, t) + \psi(n - 1, t) = \tilde{A}(t)\psi(n, t) , \quad \psi_t(n, t) = \psi(n + 1, t) ,
\]
where \( \tilde{A}(t) = H(t)A(t)H^{-1}(t) \) still belongs to \( \mathcal{G}_N[t] \). In the light of Lemma 4, \( \phi \) and \( \psi \) lead to same solutions to the Toda lattice. That also means that we can throw away \( B(t) \) from Eq. (16), and consequently the Casoratian condition (16) is simplified to
\[
\phi(n + 1, t) + \phi(n - 1, t) = A\phi(n, t) ,
\]
where \( \tilde{\psi}(n, t) = (\psi_1(n, t), \psi_2(n, t), \ldots, \psi_{N-1}(n, t))^T \).

For Set I, since \( \Gamma_{N-1} \) can be combinations of some diagonal matrices and Jordan blocks, one can directly consider that \( \Gamma_{N-1} \) is an \( (N-1) \)-order diagonal matrix with distinct eigenvalues or an \( (N-1) \)-order Jordan block. In Refs. [12–16], Set I has been discussed in detail and explicit general solutions corresponding to different \( \Gamma_{N-1} \) are given. Particularly, for \( \Gamma_{N-1} \) being a Jordan block, the general solutions can be expressed in simple forms by means of lower triangular Toeplitz matrices.[12]

For Set II, some formulas have been worked out for the solutions to the following equations[15]
\[
\begin{align*}
\psi_N(n + 1, t) + \psi_N(n - 1, t) &= \Gamma_{N-1} \tilde{\psi}(n, t) , \\
\psi_{N, t}(n, t) &= \psi_N(n + 1, t) , \\
\end{align*}
\]
where \( \tilde{\psi}(n, t) = (\psi_1(n, t), \psi_2(n, t), \ldots, \psi_{N-1}(n, t))^T \).

For differential-difference equations (34), when \( a_{NN} = 2 \), following Ref. [15] we have the following general solution,
\[
\psi_N(n, t) = \left[ \alpha(n)t + \beta(n) \right] + \int_0^t \int_0^s f(n + 1, r)e^{-\tau}drds \right] e^t ,
\]
where \( \alpha(n) \) and \( \beta(n) \) are determined by
\[
\begin{align*}
\alpha(n + 1) - \alpha(n) &= f(n + 1, 0) , \\
\beta(n + 1) - \beta(n) &= \alpha(n) .
\end{align*}
\]
Actually, this case corresponds to rational solutions.[15] When \( a_{NN} \neq 2 \), we have
\[
\psi_N(n, t) = \frac{1}{w_1 - w_2} \left[ e^{w_1t} \int_0^t f(n + 1, r)e^{-w_1r}dr - e^{w_2t} \int_0^t f(n + 1, r)e^{-w_2r}dr \right] .
\]
+ α(n)e^{w_1 t} + β(n)e^{w_2 t},
\end{equation}
where $w_1, w_2$ are defined by $w_1 + w_2 = a_{NN}$, $w_1 w_2 = 1$, and $α(n), β(n)$ are defined by
\begin{equation}
\begin{split}
w_1α(n - 1) - α(n) &= \frac{f(n, 0)}{w_2 - w_1}, \\
β(n) + α(n) &= w_1α(n - 1) + w_2β(n - 1).
\end{split}
\end{equation}
Thus we can give general solutions to Eq. (30).

4.2 Transformation of Solutions

We consider $V_n = f_{n+1}f_{n-1}/f_n^2 - 1$ as a new solution generated from the old one $V_n = g_{n+1}g_{n-1}/g_n^2 - 1$. If the form of $Γ_{N-1}$ is given out, such as diagonal or Jordan block which corresponding to $V_n = g_{n+1}g_{n-1}/g_n^2 - 1$ is $(N - 1)$-order solitons or Jordan block solutions, then obviously it is $γ$ and $a_{NN}$ to determine what kinds of new solutions ($N$-order solutions) $V_n = f_{n+1}f_{n-1}/f_n^2 - 1$ can be got. There are a lot of choices for $Γ_{N-1}$, $γ$, and $a_{NN}$, and these discussions are similar as Ref. [14]. Thus base on the form of $Γ_{N-1}$ and suitable $γ$ and $a_{NN}$, we can give the bridges between solitons, between negatons, between positons, between rational solutions, between complexitons, and between mixed solutions.

5 Conclusion

A simple analysis has been made for the Casoratian condition of the Toda lattice whose coefficient matrix consists of arbitrary matrix in $G_N$. The condition we obtained here is quite similar to the one for the bilinear Toda lattice. We proved that this condition is reasonable by imposing suitable value to the parameter $λ$ in the bilinear BT. With further discussions the condition is finally the differential-difference equation set (30) in which $Γ$ can be a quasi-canonical matrix in $G_N$. General solutions for such an equation set have been given in Refs. [12–13,19]. By choosing $Γ$ to be some diagonals or Jordan blocks, the corresponding solution to Eq. (30), i.e., the Casoratian entry vector $ψ$, can provide solitons, negatons, positons, rational solution and complexitons. Although the bilinear cannot provide any newer solutions than the bilinear equation does, in the paper we have shown that how the BT provide transformations between solitons and solitons, between negatons and negatons, between positons and positons, between rational solution and rational solution, and between complexitons and complexitons. It also admits transformations for mixed solutions. Such discussions can be generalized to other soliton equations with bilinear BT and Casoratian solutions.

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References