SYMMETRIES AND LIE ALGEBRA
OF THE DIFFERENTIAL–DIFFERENCE
KADOMSTEV–PETVIASHVILI HIERARCHY

XIAN-LONG SUN*, DA-JUN ZHANG†, XIAO-YING ZHU and DENG-YUAN CHEN
Department of Mathematics, Shanghai University, Shanghai 200444, China
*zlongs@yahoo.cn
†djzhang@staff.shu.edu.cn

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By introducing suitable non-isospectral flows, we construct two sets of symmetries for
the isospectral differential–difference Kadomstev–Petviashvili hierarchy. The symmetries
form an infinite dimensional Lie algebra.

Keywords: Non-isospectral flows; differential–difference Kadomstev–Petviashvili equation; symmetries; Lie algebra.

1. Introduction

Searching for symmetries and Lie algebraic structure is an important and interesting
topic in integrable systems.1 A variety of methods have been developed to obtain
infinitely many symmetries and their Lie algebraic structures for Lax integrable
systems,2–15 for both (1 + 1)-dimensional and high-dimensional cases. One of the
efficient ways is to use Lax representation of isospectral and non-isospectral flows
(cf. Refs. 4–7, 11, 12), rather than recursion operators, and this approach has been
extended to high-dimensional continuous integrable systems.15

In this paper, we consider the symmetries and their Lie algebra for the
differential–difference Kadomstev–Petviashvili (D∆KP) hierarchy by means of Lax
representation approach. The (isospectral) D∆KP hierarchy is derived following the
basic frame of Sato’s theory starting from a quasi-difference operator.16,17 However,
due to the lack of a neat form of discrete derivatives, the non-isospectral flows do
not have regular asymptotic properties as the continuous non-isospectral flows do.
We have to choose suitable time evolution for spectral parameter so that we can
get suitable non-isospectral flows which can be used to construct symmetries.

This paper is organized as follows. In Sec. 2, we construct isospectral and
non-isospectral D∆KP hierarchies from a quasi-difference operator. In Sec. 3, we
construct two sets of symmetries and their Lie algebra for the isospectral DΔKP hierarchy.

2. The Isospectral and Non-Isospectral DΔKP Hierarchies

Let us consider the difference analogue of a quasi-differential operator

\[ L = \Delta + u_0 + u_1 \Delta^{-1} + \cdots + u_j \Delta^{-j} + \cdots, \]

where \( u_s = u_s(n, t) = u_s(n, t_1, t_2, \ldots) \) \((s = 0, 1, 2, \ldots)\), \( t = (t_1, t_2, \ldots) \). Each \( u_s \) vanishes rapidly when \( |n| \to \infty \). \( \Delta \) denotes the forward difference operator defined by \( \Delta f(n) = (E - 1)f(n) = f(n + 1) - f(n) \) and the shift operator \( E \) is defined by \( Ef(n) = f(n + 1) \). The operators \( \Delta \) and \( E \) are connected by \( \Delta = E - 1 \) and \( \Delta \Delta^{-1} = \Delta^{-1} \Delta = 1 \). Obviously, the \( r \)-th power of \( L \) can be expressed as

\[ L^r = \sum_{j \leq r} p_{r,j}(u) \Delta^j, \]

where the coefficients \( p_{r,j}(u) \) are uniquely determined by the coordinates \( u_j \) \((j = 0, 1, 2, \ldots)\) and their differences. Here by \( u \) we denote \((u_0, u_1, \ldots)^T\). \( L^r \) can be separated into

\[ (L^r)_+ = \sum_{j=0}^n p_{r,j}(u) \Delta^j, \quad (L^r)_- = L^r - (L^r)_+, \]

where \(( \ )_+\) denotes the nonnegative part of \( \Delta \) and \(( \ )_-\) the residual part.

In general, the isospectral flows can be obtained from the compatibility of

\[ L \phi = \eta \phi, \]

\[ \phi_{t_s} = A_s \phi, \]

i.e.

\[ L_{t_s} = [A_s, L], \]

where \( \eta_{t_s} = 0, A_s = (L^s)_+ \) and obviously \( A_s \) satisfies the boundary condition

\[ A_s|_{u=0} = \Delta^s. \]

The first few explicit forms of \( A_s \) and equations given by the Lax equation (2.4a) are\(^\text{17}\)

\[ A_1 = \Delta + u_0, \]
\[ A_2 = \Delta^2 + (\Delta u_0 + 2u_0)\Delta + (\Delta u_0 + u_0^2 + \Delta u_1 + 2u_1), \]
\[ A_3 = \Delta^3 + a_1 \Delta^2 + a_2 \Delta + a_3, \]
\[ \vdots \]
with

\[ a_1 = \Delta^2 u_0 + 3\Delta u_0 + 3u_0, \]  \hspace{1cm} (2.6a)
\[ a_2 = 2\Delta^2 u_0 + 3\Delta u_0 + 3u_0^2 + 3u_0\Delta u_0 + (\Delta u_0)^2 + 3u_1 + 3\Delta u_1 + \Delta^2 u_1, \]  \hspace{1cm} (2.6b)
\[ a_3 = \Delta^2 u_0 + 5u_0u_1 + 3u_0\Delta u_0 + u_0^3 + (\Delta u_0)^2 + \Delta u_0\Delta u_1 + 3u_0\Delta u_1 \]
\[ + u_1\Delta u_0 + u_1E^{-1}u_0 + 2\Delta^2 u_1 + 3\Delta u_1 + 3u_2 + 3\Delta u_2 + \Delta^2 u_2; \]  \hspace{1cm} (2.6c)
\[ u_{0,t_1} = q_{10} = \Delta u_1, \]  \hspace{1cm} (2.7a)
\[ u_{1,t_1} = q_{11} = \Delta u_1 + \Delta u_2 + u_0u_1 - u_1E^{-1}u_0, \]  \hspace{1cm} (2.7b)
\[ u_{2,t_1} = q_{12} = \Delta u_3 + \Delta u_2 + u_0u_2 + u_1E^{-1}u_0 - u_2E^{-2}u_0 - u_1E^{-2}u_0, \]  \hspace{1cm} (2.7c)
\[ \vdots \]
\[ u_{0,t_2} = q_{20} = \Delta^2 u_1 + 2\Delta u_2 + \Delta^2 u_2 + u_1\Delta u_0 + 2u_0\Delta u_1 \]
\[ + (\Delta u_0)\Delta u_1 + u_0u_1 - u_1E^{-1}u_0, \]  \hspace{1cm} (2.8a)
\[ u_{1,t_2} = q_{21} = \Delta^2 u_1 + 2\Delta u_2 + 2\Delta^2 u_2 + 2\Delta u_3 + 2u_0\Delta u_1 + \Delta u_0\Delta u_1 \]
\[ + 2u_0u_2 + u_2\Delta u_0 + 2u_0\Delta u_2 + \Delta u_0\Delta u_2 + u_1\Delta u_0 + u_1u_0^2 + u_1^2 \]
\[ + u_1\Delta u_1 - u_1E^{-2}u_0 + u_1E^{-1}u_0 - u_1E^{-1}u_1 - u_2E^{-1}u_0 \]
\[ - u_2E^{-2}u_0 - u_1E^{-1}u_0^2, \]  \hspace{1cm} (2.8b)
\[ \vdots \]

From (2.7), we obtain

\[ u_1 = \Delta^{-1}\frac{\partial u_0}{\partial t_1}, \]  \hspace{1cm} (2.9a)
\[ u_2 = \Delta^{-1}\frac{\partial^2 u_0}{\partial t_1^2} - \Delta^{-1}\frac{\partial u_0}{\partial t_1} - E^{-1}u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1} + \Delta^{-1}\left(u_0\frac{\partial u_0}{\partial t_1}\right), \]  \hspace{1cm} (2.9b)
\[ \vdots \]

Eliminating \( u_1, u_2, \ldots \) from (2.7a), (2.8a), \ldots, one can obtain (\( u_0 = u, t_1 = y \))\(^{16,17}\)

\[ u_{t_1} = K_1 = u_y, \]  \hspace{1cm} (2.10a)
\[ u_{t_2} = K_2 = (1 + 2\Delta^{-1})u_{yy} - 2u_y + 2uu_y, \]  \hspace{1cm} (2.10b)
\[ \vdots \]

which are isospectral D\( \Delta \)KP hierarchy where Eq. (2.10b) is the well-known D\( \Delta \)KP equation.
To get the $\tau$-symmetries we need to introduce the non-isospectral DΔKP hierarchy. In this case, we set

$$\eta_{t^r} = \eta^r + \eta^{r-1}. \quad (2.11)$$

Then the Lax equation turns out to be

$$L_{t^r} = [B_r, L] + L^r + L^{r-1}, \quad (2.12a)$$

where

$$B_r = b_0 \Delta^r + b_1 \Delta^{r-1} + \cdots + b_r, \quad (r > 0), \quad (2.12b)$$

in which $b_i$ ($i = 0, 1, 2, \ldots, r$) are undetermined functions of coordinates $u_j$ ($j = 0, 1, 2, \ldots$) and their differences. $B_r$ is imposed the boundary condition

$$B_r \big|_{u=0} = t_1 \Delta^r + n \Delta^{r-1}, \quad (2.12c)$$

and then the both sides of the Lax equation (2.12a) go to zero when $u \to 0$.

The first few of $B_r$ and equations given by (2.12a) are

$$B_1 = t_1 A_1 + n, \quad (2.13a)$$
$$B_2 = t_1 A_2 + n \Delta + nu_0 + \Delta^{-1}u_0, \quad (2.13b)$$
$$B_3 = t_1 A_3 + n \Delta^2 + (2nu_0 + n\Delta u_0 + \Delta^{-1}u_0)\Delta + u_0 \Delta^{-1}u_0 + 2nu_1 + n \Delta u_0$$
$$+ n \Delta u_1 - 2u_1 - \Delta u_1 + nu_0^2 - u_0^2 + \Delta^{-1}(u_1 - u_0 + u_0^2), \quad (2.13c)$$

$$\vdots$$

$$u_{0,t_1} = t_1 q_{10} + u_0, \quad (2.14a)$$
$$u_{1,t_1} = t_1 q_{11} + 2u_1, \quad (2.14b)$$
$$u_{2,t_1} = t_1 q_{12} + u_1 + 3u_2, \quad (2.14c)$$

$$\vdots$$

One may wonder that (2.11) is a linear combination and so is the non-isospectral flow $\sigma_r$. Actually, in the Lax representation approach, we need $\sigma_r \big|_{u=0} = 0$. Suppose that we start from a general form

$$\eta_{t^r} = a\eta^\alpha + b\eta^\beta,$$

with constants $a, b$ and integers $\alpha, \beta$. Then the Lax equation is

$$L_{t^r} = [B_r, L] + aL^\alpha + bL^\beta.$$

Noting that $L \big|_{u=0} = \Delta$ and (2.12c) the right-hand side of the above equation becomes

$$-\Delta^r - \Delta^{r-1} + a\Delta^\alpha + b\Delta^\beta$$

when $u = 0$, which means we have to take $a = b = 1, \alpha = r, \beta = r - 1$ so that it vanishes. Hence we need the time evolution (2.11).
\[ u_{0,t_2} = t_1 q_{20} + n \Delta u_1 + u_0^2 - u_0 + 3 u_1 + \Delta u_1, \] (2.15a)
\[ u_{1,t_2} = t_1 q_{21} + n \Delta u_1 + n \Delta u_2 + (n + 1) u_0 u_1 + u_1 \Delta^{-1} u_0 + 2 u_1 + \Delta u_2 \\
+ 3 u_2 + (2 - n) u_1 E^{-1} u_0 - u_1 E^{-1} \Delta^{-1} u_0 + \Delta u_1, \] (2.15b)
\[ u_{0,t_3} = t_1 q_{30} - \Delta^2 u_0 + n u_0 \Delta u_1 + n \Delta (u_0 u_1) + \Delta^{-1} u_0 \Delta u_1 + n u_0 u_1 + n \Delta^2 u_1 \\
+ u_1 \Delta^{-1} u_0 - u_0 \Delta u_0 - 4 n \Delta u_1 - 2 u_1 - \Delta u_0 - \Delta u_1 + u_0 - 3 u_0^2 \\
- n u_1 E^{-1} u_0 + u_1 E^{-1} u_0 + u_0 u_1 - u_1 E^{-1} \Delta^{-1} u_0 + n \Delta^2 u_2 \\
+ 2 \Delta u_2 + 2 u_2, \] (2.16)

Here \( A_l \) and \( q_{ij} \) are described by (2.5), (2.7) and (2.8) respectively.

Then substituting (2.9) with \( t_1 = y \) into (2.14a), (2.15a) and (2.16) yields \( (u_0 = u) \),
\[ u_{t_1} = \sigma_1 = y K_1 + u, \] (2.17a)
\[ u_{t_2} = \sigma_2 = y K_2 + (1 + n) u_y + 3 \Delta^{-1} u_y + u^2 - u, \] (2.17b)

in which, \( K_s \) are given by (2.10). These equations constitute the non-isospectral hierarchy of the D\( \Delta \)KP system.

The obtained isospectral and non-isospectral D\( \Delta \)KP hierarchies can be expressed through Lax equations in the following form:
\[ L'[K_s] = [A_s, L], \] (2.18a)
\[ A_s|_{u=0} = \Delta^s; \] (2.18b)
\[ L'[\sigma_r] = [B_r, L] + L^r + L^{r-1}, \] (2.19a)
\[ B_r|_{u=0} = t_1 \Delta^r + n \Delta^{r-1}, \] (2.19b)

which we call Lax representations of flows.

3. Lie Algebra Structure of the D\( \Delta \)KP System

In this section, we begin with a discussion of Gateaux derivative concerning the quasi-difference operator. Let \( \partial_j = \frac{\partial}{\partial t_j} \) and \( \mathcal{F} \) denote a linear space constructed by all real functions \( f = f(u) \) depending on \( n, t \) and derivatives and differences of \( u \). \( f(u) \) is \( C^\infty \) differentiable with respect to \( t \) and \( n \), and vanishes rapidly when \( |n| \to \infty \). The Gateaux derivative of \( f(u) \in \mathcal{F} \) in direction \( h \in \mathcal{F} \) with respect to \( u \) is defined as
\[ f'[h] = \frac{d}{d\varepsilon} f(u + \varepsilon h)|_{\varepsilon=0}, \] (3.1)
from which \( F \) forms a Lie algebra according to the following Gateaux commutator:
\[
[f, g] = f'[g] - g'[f],
\]
where \( f, g \in F \). For a quasi-difference operator
\[
P(u) = \sum_{j \leq s} p_j(u)\Delta^j,
\]
its Gateaux derivative in direction \( h \) with respect to \( u \) is defined by
\[
P'[h] = \sum_{j \leq s} p'_j[h]\Delta^j.
\]
Besides, using
\[
(p'_j)[f]g = (p'_j)[g]f,
\]
one can get
\[
(P'[f])[g] - (P'[g])[f] = P'[[f, g]].
\]
In addition, it is easy to prove the following lemma.

**Lemma 1.** For the quasi-difference operator \( L \) defined in (2.1), \( B \) in the form (2.12) and \( X \in F \), the equation
\[
L'[X] = [B, L], \quad B|_{u=0} = 0,
\]
only admits zero solution \( X = 0, B = 0 \).

Then, from the Lax representations (2.18) and (2.19), we have the following property.

**Theorem 1.** Suppose that
\[
\langle A_s, A_r \rangle = A'_s[K_r] - A'_r[K_s] + [A_s, A_r],
\]
\[
\langle A_s, B_r \rangle = A'_s[\sigma_r] - B'_r[K_s] + [A_s, B_r],
\]
\[
\langle B_s, B_r \rangle = B'_s[\sigma_r] - B'_r[\sigma_s] + [B_s, B_r],
\]
then we have
\[
L'[[K_s, K_r]] = \langle A_s, A_r \rangle, \quad L'[[K_s, \sigma_r]] = \langle A_s, B_r \rangle,
\]
\[
L'[[\sigma_s, \sigma_r]] = \langle B_s, B_r \rangle, \quad L'[[\sigma_s, \sigma_r]] = L + (s-r)\Delta^{s+r-1} + 2(s-r)\Delta^{s+r-2} + (s-r)\Delta^{s+r-3},
\]
and
\[
\langle A_s, A_r \rangle|_{u=0} = 0,
\]
\[
\langle A_s, B_r \rangle|_{u=0} = s\Delta^{s+r-1} + s\Delta^{s+r-2},
\]
\[
\langle B_s, B_r \rangle|_{u=0} = (s-r)[t_1\Delta^{s+r-1} + (t_1 + n)\Delta^{s+r-2} + n\Delta^{s+r-3}].
\]
symmetries and Lie Algebra of Kandmachev-Peiraviashvili Hierarchy

\textbf{Proof.} We only prove the equalities \((3.9c)\) and \((3.10c)\). The others can be obtained in a similar way.

Taking the Gateaux derivative of \((2.19a)\) in the direction \(\sigma_s\) with respect to \(u\), and noting

\[ L'[\sigma_s] = [B_s, L'] + rL^{s+r-1} + rL^{s+r-2}, \quad (3.11) \]

and

\[ [[B_s, B_r], L] = [B_s, [B_r, L]] - [B_r, [B_s, L]], \quad (3.12) \]

we have

\[(L'[\sigma_s])'[\sigma_r] = [B'_s[\sigma_r], L] + [B_s, [B_r, L]] + [B_r, L^s] + [B_r, L^{s-1}] + [B_s, L'] + rL^{s+r-1} + rL^{s+r-2} + [B_r, L^{r-1}] + (r-1)L^{s+r-2} + (r-1)L^{s+r-3}.\]

\[(3.13)\]

Similarly,

\[(L'[\sigma_s])'[\sigma_r] = [B'_s[\sigma_r], L] + [B_s, [B_r, L]] + [B_s, L'] + [B_s, L^{s-1}] + [B_r, L^s] + sL^{s+r-1} + sL^{s+r-2} + [B_r, L^{s-1}] + (s-1)L^{s+r-2} + (s-1)L^{s+r-3}.\]

\[(3.14)\]

Then \((3.13)\) coupled with \((3.14)\) yields

\[(L'[\sigma_s])'[\sigma_r] - (L'[\sigma_s])'[\sigma_r] = [(B_s, B_r), L] + (s-r)(L^{s+r-1} + 2L^{s+r-2} + L^{s+r-3}), \quad (3.15)\]

which gives \((3.9c)\) by using \((3.6)\). Next, noting that \(K_s, \sigma_r \in \mathcal{F}\), i.e. \(K_s|_{u=0} = \sigma_r|_{u=0} = 0\), from \((3.8c)\) we obtain \((3.10c)\) immediately. We complete the proof. \(\square\)

With the above theorem in hand, the algebraic relation of flows \(K_s\) and \(\sigma_r\) can be derived.

\textbf{Theorem 2.} The flows \(K_s\) and \(\sigma_r\) form a Lie algebra with structure

\[ [[K_s, K_r]] = 0, \quad (3.16a) \]

\[ [[K_s, \sigma_r]] = sK_{s+r-1} + sK_{s+r-2}, \quad (3.16b) \]

\[ [[\sigma_s, \sigma_r]] = (s-r)(\sigma_{s+r-1} + \sigma_{s+r-2}), \quad (3.16c) \]

where \(s, r \geq 1\) and we set \(K_0 = \sigma_0 = 0\).

\textbf{Proof.} In the light of \((3.7)\) only admitting zero solution, \((3.9a)\) coupled with \((3.10a)\) possesses the same property as well, which means \((3.16a)\) holds.

Next, taking

\[ \theta = [[K_s, \sigma_r]] - sK_{s+r-1} - sK_{s+r-2}, \quad (3.17a) \]

\[ \hat{A} = [A_s, B_r] - sA_{s+r-1} - sA_{s+r-2}, \quad (3.17b) \]
it then follows from (3.9b), (3.10b) and the isospectral Lax representation (2.18) that
\[ L'[\theta] = [\hat{A}, L], \quad \hat{A}|_{u=0} = 0, \] (3.18)
which has only zero solution \( \theta = 0 \) and \( \hat{A} = 0 \), and that means (3.16b) is true.

Similarly, taking
\[ \omega = \left[ \sigma_s, \sigma_r \right] - (s-r)(\sigma_{s+r-1} + s\sigma_{s+r-2}), \] (3.19a)
\[ \hat{B} = \langle B_s, B_r \rangle - (s-r)(B_{s+r-1} + B_{s+r-2}), \] (3.19b)
and noting that \( \hat{B}|_{u=0} = 0 \) together with (2.19), (3.9c) and (3.10c), we then have
\[ L'[^{\omega}] = [\hat{B}, L], \quad \hat{B}|_{u=0} = 0. \] (3.20)
Hence we get \( \omega = 0 \) and \( \hat{B} = 0 \), which shows that (3.16c) is also correct. Thus we complete the proof.

Based on Theorem 2, the symmetries and their algebraic structure for the isospectral D∆KP hierarchy \( u_t = K_s \) can be derived immediately.

**Theorem 3.** The isospectral D∆KP hierarchy \( u_t = K_s \) can have two sets of symmetries, \( K \)-symmetries \( \{K_l\} \) and \( \tau \)-symmetries, \( \tau^s_r = st_s K_{s+r-1} + st_r K_{s+r-2} + \sigma_r \) (\( l = 1, 2, \ldots, r = 1, 2, \ldots \)), which form a Lie algebra with structure
\[ [[K_l, K_r]] = 0, \] (3.21a)
\[ [[K_l, \tau^s_r]] = l(K_{l+r-1} + K_{l+r-2}), \] (3.21b)
\[ [[\tau^s_l, \tau^s_r]] = (l-r)(\tau^s_{l+r-1} + \tau^s_{l+r-2}), \] (3.21c)
where \( l, r, s \geq 1 \) and we set \( K_0 = \tau^0_0 = 0 \). Especially for the D∆KP equation (2.10b), its symmetries are \( K_l \) and \( \tau_r = 2tK_{r+1} + 2tK_r + \sigma_r \).

We end this section by the following two remarks. First, the new time-dependence (2.11) of the spectral parameter \( \eta \) leads to the new algebra structure (3.21), which is different from the centerless Kac–Moody–Virasoro algebra (cf. Ref. 19) of the D∆KP equation given in Refs. 17 and 20, and also different from the centerless Kac–Moody–Virasoro algebra of the KP hierarchy obtained in Ref. 15. Besides, \( \{K_1, K_2, \tau^2_1\} \) compose a subalgebra. This agrees with the symmetry algebra of the D∆KP equation obtained in Ref. 21 where \( \tau^2_1 \) provides an invariability for the D∆KP equation under a combined Galilean-scallar transformation, and now \( \tau^2_1 \) has got its clear context in the Lax representation approach. The second remark is on the relation between the D∆KP equation and the KP equation. In fact, the D∆KP equation was originally proposed by Date et al.\(^{22}\) It was derived from a bilinear identity (discretized by partially imposing Miwa’s transformation on continuous exponential functions) which is related to the KP hierarchy. However, since in the discrete exponential the discrete variables (e.g. \( n, m, l \)) appear symmetrically and do not represent dispersion relation as in the continuous one, therefore there
are (sometimes complicated) variable combination and transformation involved in the continuous limit procedure. There have been many results on the D∆KP equation, such as bilinear form, Sato’s approach, Casoratian solutions, gauge transformation and double Casoratian solutions, and also symmetries in the present paper. The relations between these results and those of the KP equation will be investigated in detail elsewhere in terms of continuous limit.

4. Conclusion

In this paper, by introducing suitable time-dependence \( \eta_r = \eta^r + \eta^{r-1} \) for the spectral parameter \( \eta \), we obtained non-isospectral D∆KP flows \( \{ \sigma_r \} \) which satisfy \( \sigma_r |_{\eta=0} = 0 \). This enables us to construct \( K \)-symmetries and \( \tau \)-symmetries for the isospectral D∆KP hierarchy through the Lax representation approach. The obtained symmetries are proved to form a Lie algebra.

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