Bilinear approach for a symmetry constraint of the modified KdV equation

Jian-bing Zhang\textsuperscript{a,*}, Da-jun Zhang\textsuperscript{b}, Qing Shen\textsuperscript{b}

\textsuperscript{a}School of Mathematics Science, Xuzhou Normal University, Xuzhou, Jiangsu 221116, PR China
\textsuperscript{b}Department of Mathematics, Shanghai University, Shanghai 200444, PR China

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abstract

The paper investigates the mKdV equation with the potential under symmetry constraint through bilinear approach, i.e., Hirota method and Wronskian technique. We show that the potential can be a summation of squares of wave functions and these wave functions can precisely be described as Wronskians.

1. Introduction

Both bilinear approach\textsuperscript{[1]} and symmetry method\textsuperscript{[2]} provide efficient ways for finding analytic solutions to soliton equations. The former makes use of Hirota’s bilinear form and solves bilinear equations through Hirota method\textsuperscript{[1,3]} or Wronskian technique\textsuperscript{[4]}. In the last few years there have been many works on bilinear approach. In 2002, Wenxiu Ma considered the Wronskian and Casoratian solutions to integrable equations under the combined conditions\textsuperscript{[5,6]}. In 2010, Jian-bing Zhang, Da-jun Zhang and Deng-yuan Chen solved the KdV equation under Bargmann constraint via bilinear approach\textsuperscript{[7]}. In 2011, the linear superposition principle was used to solve Hirota bilinear equations\textsuperscript{[8]}. More recently Wenxiu Ma successfully obtained the combined Wronskian solutions to the 2D Toda molecule equation\textsuperscript{[9]}.

In the later a symmetry is related to a simple-parameter transformation group for the corresponding equation, and in principle a symmetry can lead to a similar reduction by a symmetry constraint. The symmetry constraints for 1+1 dimensional integrable systems were presented for the first time in\textsuperscript{[10]} and the symmetry constraint which we will discuss is a reduced case of the binary symmetry constraint in\textsuperscript{[11]}. It is important to note that symmetry constraints may not generate exact solutions to soliton equations\textsuperscript{[12]}.

In the paper we will consider a symmetry constraint for the modified KdV (mKdV) equation and solve the constraint system by bilinear approach.

The mKdV equation is

\[ u_t + 6u^2u_x + u_{xxx} = 0, \tag{1.1} \]

with the Lax pair

\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
= 
\begin{pmatrix}
-\dot{\lambda} & u \\
-u & \dot{\lambda}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix},
\]

\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
= 
\begin{pmatrix}
4\lambda^3 + 2u^2\dot{\lambda} & -4u\dot{\lambda}^2 + 2u\dot{\lambda} - u_{xx} - 2u^3 \\
4u\dot{\lambda} + 2u\dot{\lambda} + u_{xx} + 2u^2 & 4\dot{\lambda}^3 - 2u\dot{\lambda}^2
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}. \tag{1.2}
\]
It is quite easy to verify that \( u \) and \((\phi_1^2 + \phi_2^2)\) are the symmetries of the mKdV Eq. (1.1) where \( \phi_1 \) and \( \phi_2 \) satisfy the Lax pair (1.2). In the paper we consider the following symmetry constraint

\[
\sigma = u_x - \sum_{j=1}^{N} (\phi_i^2_j + \phi_2^2_j)_x = 0, \tag{1.3}
\]

where each \((\phi_i^2_j, \phi_2^2_j)^T\) solves the Lax pair (1.2) with \( \lambda = i \), i.e., satisfying

\[
(\phi_{ij})_x = \left[ \begin{array}{c} -\lambda_{ij} \ 0 \\ -u \end{array} \right] (\phi_{ij}),
\]

\[
(\phi_{2ij})_x = \left[ \begin{array}{c} 4\lambda_{ij}^2 + 2u^2 \lambda_{ij} \\ -4\lambda_{ij}^2 + 2u \lambda_{ij} - u_{xx} - 2u^3 \end{array} \right] (\phi_{2ij}). \tag{1.4}
\]

Thus (1.3), (1.4) and the mKdV Eq. (1.1) itself make up a symmetry constraint system for (1.1). If we replace (1.3) by

\[
u = \sum_{j=1}^{N} (\phi_i^2_j + \phi_2^2_j), \tag{1.5}
\]

then we by direct substitution can find (1.5) solves the mKdV Eq. (1.1). Consequently, the symmetry constraint system turns out to consist of only (1.5) and (1.4).

On other backgrounds of (1.5), if we consider the AKNS-ZS spectral problem

\[
(\phi_{ij})_x = \left[ \begin{array}{c} -\lambda_{ij} \ q \\ r \end{array} \right] (\phi_{ij}),
\]

then from the Gel’fand–Levitan–Machenko equation in the inverse scattering transform procedure, it can be derived that [13]

\[
q = \sum_{j=1}^{N} \phi_{ij}^2, \quad r = -\sum_{j=1}^{N} \phi_{2ij}^2. \tag{1.7}
\]

Obviously, under the reduction \( q = -r = u \), (1.5) is a linear combination of (1.7). (1.7) is called Bargmann-type map, by which one can nonlinearize a Lax pair and get finite-dimensional Hamiltonian systems [13–19]. Besides, such kind of constraint is also related to soliton equations with sources [20]. In addition, it is well known that the Bäcklund transformation or bilinear Bäcklund transformation is also closely related to Lax pairs [21,22]. Since the symmetry constraint system is just a combination of the Lax pair (1.4) and the constraint (1.5), this combination might be solvable by some transformation approach.

Bilinear method has been shown to be efficient for finding solutions for both continuous systems [1] and fully discrete systems (cf. [23,24]). In recent years, solutions in Wronskian form received new understanding and Wronskian can include more kinds of solutions beyond solitons [25–27]. Besides, some Wronskian formulae related to soliton equations with sources [28,29] will be helpful for our discussion in the paper.

In the following we will solve the symmetry constraint system (1.5) and (1.4) by Hirota method in Section 2 and Wronskian technique in Section 3.

2. Hirota method

In the section we first transform (1.5) and (1.4) into bilinear forms and then solve them by means of Hirota method [3,1]. By the transformation [21,28]

\[
\phi_{ij} = \frac{g_j}{f}, \quad \phi_{2ij} = i \left( \frac{g_j}{f} \right), \quad (j = 1, 2, \ldots N), \tag{2.1a}
\]

\[
u = i \left( \ln f \right)_x, \tag{2.1b}
\]

where \( f, g_j \) means complex conjugate of \( f, g_j \) and \( i \) is the imaginary unit, (1.5) and (1.4) can be transformed into the following bilinear form

\[
i D_x f \cdot \bar{f} = 4 \sum_{j=1}^{N} \bar{g}_j g_j, \tag{2.2a}
\]

\[
D_x^2 f \cdot \bar{f} = 0, \tag{2.2b}
\]

\[
D_x f \cdot \bar{f} = -\lambda_j g_j \bar{f}, \tag{2.2c}
\]

\[
(D_x + D_x^2 + 3\lambda_j D_x^2) g_j \cdot f = 0, \tag{2.2d}
\]

where \( \lambda = i \).
where we have made use of [21]
\[ 2(\ln f)_{xx} = u^2 + iux, \]
(2.3)

if (2.1b) and (2.2b) hold. \( D \) is the well-known Hirota's bilinear operator defined by [1,3]
\[ D^m D^n a(t, x) \cdot b(t, x) = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t + s, x + y) b(t - s, x - y) \big|_{s=0, y=0}. \quad m, n = 1, 2, \ldots. \]

Then, inserting the expansions
\[ f = 1 + f^{(2)} e^{2i} + f^{(4)} e^{4} + \cdots, \]
\[ g_j = g^{(1)}_j e^{2i} + g^{(3)}_j e^{4} + g^{(5)}_j e^{6} + \cdots \]
into (2.2), we can get the \( N \)-soliton solution of the mKdV equation from (2.2), by taking \( \lambda_j = -k_j \) and
\[ f^{(2)} = \sum_{j=1}^N e^{2i \xi_j}, \quad g^{(1)}_j = \sqrt{k_j} e^{2i \xi_j}, \quad \xi_j = k_j x + \omega_j t + \varepsilon_j^{(0)}, \]
(2.5)

for \( j = 1, 2, \ldots, N \), where \( k_j, \omega_j \) and \( e^{\varepsilon_j^{(0)}} \) are all real constants.

In fact, when \( N = 1 \), employing the standard Hirota's approach, we find that (2.4) can be truncated by taking
\[ \omega_1 = -4k_1^3, \quad f^{(2)} = e^{2i \xi}, \quad g^{(1)}_1 = f^{(2)} = 0, \quad k = 2, 3, \ldots. \]

Then from (2.1b) we can obtain 1-soliton solution of the mKdV equation as
\[ u = 2k \text{sech}^2 \xi, \]
(2.6)

where we have taken \( \varepsilon = 1 \) in (2.1b). Similarly, when \( N = 2 \), we have truncated solutions of (2.2) as
\[ f = 1 + i(e^{2i \xi_1} + e^{2i \xi_2}) - e^{2i(\xi_1 + \xi_2 + \lambda_1)}, \]
\[ g_1 = \sqrt{k_1} e^{i \xi_1} + i \sqrt{k_1} e^{i(\xi_1 + \lambda_1 + \lambda_2)}, \]
\[ g_2 = \sqrt{k_2} e^{i \xi_2} - i \sqrt{k_2} e^{i(\xi_2 + \lambda_1 + \lambda_2)}, \]
with \( \omega_1 = -4k_1^3 \), \( \omega_2 = b_j^k - k_j \) for \( l, j = 1, 2 \). Here we also have taken \( \varepsilon = 1 \) in (2.4), and the 2-soliton solution is provided by (2.1b). We can continue the procedure to get 3-solitons and 4-solitons, and for the general \( N \), we have
\[ f = \sum_{\mu=0,1} \exp \left\{ \sum_{j=1}^N 2 \mu_j \left( \xi_j + \frac{\pi}{2} i \right) + \sum_{1 < j < l} \mu_j \mu_l a_{jl} \right\}, \]
(2.7a)

\[ g_h = \sqrt{k_h} \sum_{\mu=0,1} \exp \left\{ \mu_h \zeta_h + \sum_{j=1}^{h-1} \mu_j \left( 2 \xi_j + \frac{\pi}{2} i \right) + \sum_{j=h+1}^N \mu_j \left( 2 \xi_j + \frac{\pi}{2} i \right) - \frac{\pi}{2} i \right\} + \sum_{1 < j < l} \mu_j \mu_l a_{jl} \} \]
(2.7b)

\[ \omega_j = -4k_j^3, \quad \lambda_j = -k_j, \quad e^{\varepsilon_j} = \frac{k_j - k_l}{k_j + k_l}, \]
(2.7c)

where the sum over \( \mu = 0, 1 \) refers to each of the \( \mu_j \), \( j = 1, 2, \ldots, N \) and \( \{k_j\} \) satisfy \( 0 < k_1 < k_2 < \cdots < k_N \).

3. Wronskian technique

Next we consider the potential \( u \) and wave functions \( \{\phi_1, \phi_2\} \) in Wronskian form. An \( N \times N \) Wronskian is defined by
\[ W = \begin{vmatrix} \phi_1 & \phi_1(1) & \ldots & \phi_1(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N & \phi_N(1) & \ldots & \phi_N(N-1) \end{vmatrix} = |\varphi, \partial \varphi, \ldots, \partial^{N-1} \varphi| = |0, 1, 2, \ldots, N-1| = |N-1|, \]
(3.1)

where \( \phi_j^{(k)} = \frac{\partial^k \phi_j}{\partial \tau^k}, \partial = \frac{\partial}{\partial \tau} \) and \( \varphi = (\phi_1, \phi_2, \ldots, \phi_N)^T \).

For the solutions in Wronskian form to the bilinear system (2.2), we have the following result.
**Theorem 1.** The Wronskians

\[ f = |N - 1|, \quad g_j = c_j |N - 2|, \quad (j = 1, 2, \ldots, N), \]  

(3.2)

solve the bilinear system (2.2) provided the Wronskian entries satisfy

\[ \varphi_{jx} = -k_i \phi_j, \]
\[ \varphi_{jt} = -4 \varphi_{jxx}, \]  

(3.3a, 3.3b)

where \( \tau_j = (\delta_{j1}, \delta_{j2}, \ldots, \delta_{jN})^T \), and \( c_j \) is some constant related to \( \varphi_j \).

Since (2.2c) and (2.2d) are just the bilinear BT of the mKdV equation [21] and Wronskian solutions for the BT have been given in Ref. [30], and (2.2b) has also been verified under the condition (3.3a) [28,29], we only need to focus on (2.2a). Before we give a detailed proof, let us first consider some lemmas.

**Lemma 1**

\[ \prod_{j=1}^{N} \left( k_i^2 - k_j^2 \right) = \sum_{m=0}^{N-1} (-1)^m k_i^{(N-m-1)} \left( \sum_{\substack{j_1, j_2, \ldots, j_m \in \{1, 2, \ldots, N\} \setminus \{i\} \atop j_1 < j_2 < \ldots < j_m}} k_{j_1}^2 k_{j_2}^2 \ldots k_{j_m}^2 \right). \]  

(3.4)

**Lemma 2.** Suppose that \( \varphi_j \) satisfies (3.3), then

\[ C_j = \nabla \varphi_j \nabla f_j - \varphi_j \nabla f_j \]  

is a constant.

In fact, in the light of (3.3), it is easy to show that

\[ \frac{\partial C_j}{\partial N} = \frac{\partial C_j}{\partial t} = 0. \]

**Lemma 3.** Suppose that \( A_{jN} \) is the cofactor of \( f = |N - 1| \) with respect to the \( j \)th row and \( N \)th column, and the entries \( \{ \varphi_j \} \) satisfy (3.3a). We define

\[ P[m] = \sum_{\substack{j_1, j_2, \ldots, j_m = 1 \atop j_i \neq j_j \forall i \neq j}}^{N} \varphi_{j_1}^{(N-m)} \varphi_{j_2}^{(N-m-1)} A_{j_1, N} A_{j_2, N} A_{j_3, N} \ldots (1 \leq m \leq N - 2), \]

(3.6a)

\[ R[m] = \sum_{\substack{j_1, j_2, \ldots, j_m = 1 \atop j_i \neq j_j \forall i \neq j}}^{N} \left( \prod_{j=1}^{m} k_i^2 \right) \varphi_{j_1}^{(N-m)} \varphi_{j_2}^{(N-m-1)} A_{j_1, N} A_{j_2, N} A_{j_3, N} \ldots (1 \leq m \leq N - 1), \]

(3.6b)

and specially,

\[ P[0] = \sum_{\substack{j_1, j_2 = 1 \atop j_i \neq j_j \forall i \neq j}}^{N} \varphi_{j_1}^{(0)} \varphi_{j_2}^{(0)} A_{j_1, N} A_{j_2, N}, \]

(3.7a)

\[ R[0] = \sum_{j=1}^{N} \varphi_{j}^{(0)} A_{j, N}, \]

(3.7b)

then we have

\[ P[m] = -\frac{1}{m+1} [P[m+1] + R[m+1]], \quad (0 \leq m \leq N - 3), \]

(3.8)

\[ P[N-2] = -\frac{1}{N-1} R[N-1]. \]

(3.9)

Besides, by successively using the above formulae we can have

\[ P[0] = -\sum_{j=1}^{[\frac{N}{2}]} \frac{1}{(2j-1)!} R[2j-1] + \sum_{j=1}^{[\frac{N-1}{2}]} \frac{1}{(2j)!} R[2j]. \]

(3.10)
Proof. Noting that
\[
\sum_{j=1}^{N} \phi_j^{(N-s)} A_{jn} = \begin{cases} |N-1|, & s=1 \\ 0, & s=2, 3, \ldots, N, \end{cases}
\]  
we have
\[
\phi_j^{(N-s)} A_{jn} = - \sum_{l=1}^{N} \phi_l^{(N-s)} A_{ln}, \quad (s=2, 3, \ldots, N).
\]  
For \( P[0] \) defined by (3.7), by virtue of \( \phi_j^{(N)} = k_j^2 \phi_j^{(N-2)} \) we have
\[
P(0) = \sum_{j=1}^{N} \phi_j^{(N)} \phi_j^{(N-1)} \overline{A}_{jn} A_{jn}.
\]  
Further, by means of (3.12),
\[
P(0) = - \sum_{j=1}^{N} k_j^2 \phi_j^{(N-1)} A_{jn} \sum_{m=1}^{N} \phi_m^{(N-2)} \overline{A}_{mn} = - \sum_{j=1}^{N} \left( \sum_{m \neq j} + \sum_{j \neq m} \right) k_j^2 \phi_m^{(N-2)} \phi_j^{(N-1)} \overline{A}_{mn} A_{jn}
\]  
\[
= - \sum_{j=1}^{N} k_j^2 \phi_j^{(N-2)} \phi_j^{(N-1)} \overline{A}_{jn} A_{jn} = -(R[1] + \overline{P}[1]).
\]  
Similarly,
\[
P[1] = \sum_{j_1,j_2,j_3=1}^{N} k_{j_1}^2 k_{j_2}^2 k_{j_3}^2 \phi_j^{(N-2)} \phi_j^{(N-3)} \overline{A}_{jn} A_{jn} = \sum_{j_1,j_2,j_3=1}^{N} k_{j_1}^2 k_{j_2}^2 k_{j_3}^2 \phi_j^{(N-3)} \overline{A}_{jn} A_{jn} = - \sum_{j=1}^{N} k_j^2 \phi_j^{(N-2)} \phi_j^{(N-3)} \overline{A}_{jn} A_{jn}
\]  
\[
= - \left( \sum_{j=1}^{N} \sum_{j_1 \neq j} + \sum_{j_2 \neq j} + \sum_{j_3 \neq j} \right) k_j^2 \phi_j^{(N-2)} \phi_j^{(N-3)} \overline{A}_{jn} A_{jn}
\]  
\[
= - \left( \sum_{j=1}^{N} \sum_{j_1 \neq j} + \sum_{j_2 \neq j} + \sum_{j_3 \neq j} \right) k_j^2 \phi_j^{(N-2)} \phi_j^{(N-3)} \overline{A}_{jn} A_{jn} = -(P[1] + R[2] + P[2]),
\]  
which further yields
\[
P[1] = - \frac{1}{2} (R[2] + P[2]).
\]  
As examples we give a detailed proof for \( P[0] \) and \( P[1] \). Other formulae can be derived in a quite similar way and here we skip the proof. □

Lemma 4. Suppose that \( A_{jn} \) is the cofactor of \( f = |N-1| \) with respect to the \( j \)th row and \( N \)th column, the entries \( \{ \phi_j \} \) satisfy (3.3a), \( P[m] \) and \( R[m] \) are defined as in Lemma 3. Then, for any positive number \( N \), we have
\[
f_{f_S} f_{f_S} = \sum_{m=0}^{N-1} \frac{1}{m!} (R[m] - R[m]).
\]  
Proof. We start from
\[
f_{f_S} = \frac{|N-2,N||N-1|}{|N-2|}.
\]  
Expanding \( |N-2,N| \) and \( |N-1| \) by its \( N \)th column, we have
the same bilinear symmetry constraint and the Wronskian entry invariant solutions for the mKdV equation. We also developed a procedure to achieve the Wronskian verification for the in (3.2).

So comparing (3.19) with (2.2a) we can take

\[ \phi_j^{(2s)} = k_j^{2s} \phi_j, \quad (s = 0, 1, \ldots) \]  

and based on Lemmas 1, 2 and 4, we can easily find

\[ i(f\bar{f} - \bar{f}f) = \begin{cases} 
\sum_{j=1}^{N} C_j \left[ \prod_{l=1}^{j-1} \left( k_l^2 - k_j^2 \right) \right] \bar{\Lambda}_{jn} \Lambda_{n}, & (N \text{ is an odd number}), \\
-\sum_{j=1}^{N} C_j \left[ \prod_{l=1}^{j-1} \left( k_l^2 - k_j^2 \right) \right] \Lambda_{jn} \bar{\Lambda}_{n}, & (N \text{ is an even number}).
\end{cases} \]  

The final thing to complete the proof is to give reasonable (or consistent) \( C_j \) and \( c_j \). To get \( N \)-soliton solution we need to take

\[ \phi_j = i a_j e^{i \xi_j} + \left( -1 \right)^{N-j-1} b_j e^{-i \xi_j}, \]  

where \( \xi_j \) is defined as in Section 2, \( a_j \) and \( b_j \) are real and positive. From (3.5) we find

\[ C_j = 4ik_j a_j b_j \left( -1 \right)^{N-j-1}. \]  

So comparing (3.19) with (2.2a) we can take

\[ c_j = 2 \sqrt{k_j a_j b_j} \prod_{l=1}^{j-1} \left( k_l^2 - k_j^2 \right) \frac{1}{N} \prod_{l=j+1}^{N} \left( k_l^2 - k_j^2 \right) \]  

in (3.2).

To sum up, we have solved a symmetry constraint system by bilinear approach and got \( N \)-soliton solutions as group invariant solutions for the mKdV equation. We also developed a procedure to achieve the Wronskian verification for the bilinear symmetry constraint Eq. (2.2a). This procedure is general and might be shared by those equations which have the same bilinear symmetry constraint and the Wronskian entry \( \phi_j \) satisfying (3.3a).

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