Lump Solutions of Kadomtsev–Petviashvili I Equation in Non-uniform Media

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Abstract N-lump solutions of the Kadomtsev–Petviashvili I equation in non-uniform media are derived through the inverse scattering transform. The obtained solutions describe lump waves with time-dependent amplitudes and velocities. Dynamics of 1-lump wave and interactions of two lump wave are illustrated.

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1 Introduction

A typical nonlinear phenomenon modeled by the Kadomtsev–Petviashvili I (KPI) equation

\[ u_t = u_{xxx} + 6uu_x - 3\partial^{-1}u_{yy} \]  (1)
is lump waves.\(^1\) Solutions for lumps are real rational solutions, which were derived from the inverse scattering transform (IST)\(^2\) (also cf. [3]) as well as from classical multisolitons by imposing long-wave limits.\(^3\) Usually when a soliton travels in a non-uniform media its amplitude and velocity will change along with time. Such solitons are often modeled by non-isospectral soliton equations, which are related to time-dependent spectral parameters in Lax pairs.\(^4\) This is because a spectral parameter is usually directly related to the amplitude and velocity of a soliton. As their isospectral partners, non-isospectral equations can also be dealt with via many classical approaches, such as IST, bilinear method, Darboux transformation and so on (cf. [6–7, 10–13], etc.).

In this paper we will investigate a non-isospectral KPI equation

\[ u_t = \frac{1}{4}u(u_{xxx} + 6uu_x - 3\partial^{-1}u_{yy}) - \frac{1}{2}xu_y - \partial^{-1}u_y. \]  (2)

This equation is Lax integrable and shares the same spectral problem with (1) but the spectral parameter \(k\) is related to time by \(k_t = k^2/2\) (see (5) and (41)). We will solve the equation via IST and derive its \(N\)-lump solutions. Since (2) and (1) share same direct scattering problem, we will mainly investigate the inverse scattering procedure, in which we derive time dependence of all scattering data in detail. We will also investigate dynamics of the obtained solutions. They describe lump waves moving with time-dependent amplitudes and velocities.

The paper is organized as follows. In Sec. 2 we recall the direct scattering problem. In Sec. 3 we determine time dependence of scattering data, and finally in Sec. 4 we give \(N\)-lump solutions and investigate non-isospectral dynamics.

2 Direct Scattering Problem

Let us briefly recall the direct scattering problem (cf. [2–3]). We start from the Lax pair of the non-isospectral KPI equation (2), which is

\[ i\phi_y + \phi_{xx} + u\phi = 0, \]  (3a)

\[ i\phi_t = \frac{1}{4}y(4\phi_{xxx} + 6u\phi_x + (3u_x - 3i\partial^{-1}u_y)\phi) + \frac{1}{2}x(\phi_{xx} + 2u\phi) + \frac{1}{2}\phi_x + \frac{1}{2}\phi \]
\[ + \frac{1}{4}(\partial^{-1}u)\phi + \alpha(k)\phi, \]  (3b)

where \(i\) the imaginary unit, \(\partial^{-1} = (f^\infty_x - f^{-\infty}_x)/2\), \(u\) and its \(x\)-derivatives decrease rapidly to zero when \(|x|\) and \(|y|\) → \(\infty\), and \(\alpha(k)\) is some function of the spectrum \(k\) and will be determined in the inverse scattering procedure. Here in (3b) \(k\) is a real number but it can be extended to the whole complex plane in direct scattering discussions. In the direct scattering problem the time \(t\) is fixed and is a dummy variable, thus it will be suppressed in the notations of this section. By the transformation

\[ \phi = \psi e^{i(kx-k^2y)}, \]  (4)

one can introduce spectrum in the spatial part (3a) and then this part is written in terms of \(\psi\) as

\[ i\psi_y + \psi_{xx} + 2iku\psi_x = -u\psi. \]  (5)

This equation can have two sets of Jost solutions: \(\psi^\pm(x, y, k)\) and \(\psi^\pm(x, y, k, l)\), which are respectively expressed through

\[ \psi^\pm(x, y, k) = \frac{1}{\sqrt{2\pi}} \int d^2k e^{ik(x-x_0)} e^{i\pm k^2y}
\]  (6a)

\[ \psi^\pm(x, y, k, l) = \frac{1}{\sqrt{2\pi}} \int d^2k e^{ik(x-x_0)} e^{i\pm k^2y}
\]  (6b)
\[
\psi^\pm(x, y, k) = 1 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^\pm(x - x', y - y', k)u(x', y')\psi^\pm(x', y', k)\,dx'\,dy',
\]

\[
\psi^\pm(x, y, k, l) = e^{\gamma(x, y, k, l)} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^\pm(x - x', y - y', k)u(x', y')\psi^\pm(x', y', k, l)\,dx'\,dy',
\]

where

\[
\gamma(x, y, k, l) = i(l - k)x - i(l^2 - k^2),
\]

\[
G^\pm(x - x', y - y', k) \text{ are two Green functions governed by}
\]

\[
iG_y + G_{xx} + 2ikG_x = -\delta(x - x', y - y'),
\]

and expressed as

\[
G^\pm(x - x', y - y', k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^\pm(y - y', \mu, k) e^{i\mu(x - x')}\,d\mu,
\]

where

\[
g^\pm(y - y', \mu, k) = e^{-i\mu(\mu + 2k)(y - \eta)}[H(y - y')H(\mp \mu) - H(y' - y)H(\pm \mu)],
\]

and \(H(\mu)\) is the Heaviside function. \(\psi^+(x, y, k)\) and \(\psi^-(x, y, k)\) follow the same normalized asymptotics

\[
\psi^\pm(x, y, k, l) \sim 1, \quad |k| \to \infty,
\]

and they are respectively analytic in upper and lower half \(k\)-plane except finite simple poles and continuous up to the real axis \(\text{Im}k = 0\). \(\psi^\pm(x, y, k, l)\) have one more complex freedom \(l\) than \(\psi^\pm(x, y, k)\). They have same analytical and continuous properties as \(\psi^\pm(x, y, k)\) but different asymptotics:

\[
\psi^\pm(x, y, k, l) \sim e^{\gamma(x, y, k, l)}, \quad |k| \to \infty.
\]

In what follows we collect all the scattering data. There is a jump between \(\psi^+(x, y, k)\) and \(\psi^-(x, y, k)\) on the real axis \(\text{Im}k = 0\), which is\(^{[2]}\)

\[
\psi^+(x, y, k) - \psi^-(x, y, k) = \int_{-\infty}^{\infty} T(k, l)\psi^-(x, y, k, l)\,dl,
\]

where

\[
T(k, l) = \frac{i}{2\pi} \text{sgn}(k - l) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x', y')\psi^+(x', y', k) e^{-\gamma(x', y', k, l)}\,dx'\,dy'.
\]

Besides, in lower \(k\)-plane \(\psi^-(x, y, k, l)\) and \(\psi^-(x, y, k)\) are related through

\[
\psi^-(x, y, k, l) = e^{\gamma(x, y, k, l)}\psi^-(x, y, k) - \int_{l}^{k} R(h, l) e^{\gamma(x, y, k, h)}\psi^-(x, y, h)\,dh,
\]

with

\[
R(k, l) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x', y')\psi^-(x', y', k, l)\,dx'\,dy'.
\]

These two equations combine together and yield a scattering equation

\[
\psi^+(x, y, k) - \psi^-(x, y, k) = \int_{-\infty}^{\infty} f(k, l) e^{\gamma(x, y, k, l)}\psi^-(x, y, l)\,dl,
\]

where \(f(k, l)\) is one of scattering data, satisfying

\[
f(k, l) = \frac{i}{2\pi} \text{sgn}(k - l) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x', y')\psi^-(x', y', k, l)\,dx'\,dy'.
\]

Suppose that all the discrete eigenvalues of the homogeneous part of the integral equation (6) are simple and denoted by \(k_j^\pm, j = 1, 2, \ldots, n^\pm\), where \(\text{Im}k_j^+ > 0\) and \(\text{Im}k_j^- < 0\). Actually, if \(u\) is real from (5) one can find \(k_j^- = k_j^+\) as well as \(n^- = n^+\), where \(*\) means complex conjugate. \(\{k_j^\pm\}\) contribute a second part of scattering data. Let \(\psi_j^+(x, y)\) be the solutions of the homogenous integral equation of (6) w.r.t. \(k = k_j^\pm\). By means of them the Jost solutions \(\psi^\pm(x, y, k)\) can be represented by\(^{[3,14–15]}\)

\[
\psi^\pm(x, y, k) = 1 + \sum_{j=1}^{n^\pm} \frac{i\psi_j^+(x, y)}{k - k_j^\pm} + \theta^\pm(x, y, k),
\]

where

\[
\theta^\pm(x, y, k, l) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^\pm(x - x', y - y', k)u(x', y')\theta^\pm(x', y', k, l)\,dx'\,dy',
\]

\[
\theta^\pm(x, y, k, l) = e^{\gamma(x, y, k, l)}\theta^\pm(x, y, k) - \int_{l}^{k} R(h, l) e^{\gamma(x, y, k, h)}\theta^\pm(x, y, h)\,dh,
\]

with

\[
R(k, l) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x', y')\theta^\pm(x', y', k, l)\,dx'\,dy'.
\]
where \( \theta^\pm(x, y, k) \) tend to 0 as \( |k| \to \infty \) and are analytic respectively in upper and lower half \( k \)-plane and continuous up to real axis. Besides, \( \psi_j^\pm(x, y) \) satisfy

\[
1 \mp \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x', y') \psi_j^\pm(x', y') \, dx' \, dy' = 0.
\]  

Next, define

\[
\omega_j^\pm(x, y, k) = \psi_j^\pm(x, y, k) - \frac{i \psi_j^\pm(x, y)}{k - k_j^\pm}, \quad j = 1, 2, \ldots, n^\pm.
\]  

Using (18) one can find that \( \omega_j^+(x, y, k_j^+) - (x - 2k_j^+ y) \psi_j^+(x, y) \) also solve the homogenous part of the integral equation (6) w.r.t. \( k = k_j^+ \), which means there exist constants \( \beta_j^+ \) such that \( \omega_j^+(x, y, k_j^+) - (x - 2k_j^+ y) \psi_j^+(x, y) = \beta_j^+ \psi_j^+(x, y) \). Thus, all the scattering data are ready:

\[
\{ (k,l), \beta_j^+ \}, \quad j = 1, 2, \ldots, n^+\}
\]  

where \( (k,l) \) is defined by (16b) and each \( \beta_j^+ \) is governed by

\[
\lim_{k \to k_j^+} \omega_j^+(x, y, k) = (x - 2k_j^+ y + \beta_j^+) \psi_j^+(x, y),
\]

or, equivalently by

\[
\lim_{k \to k_j^+} \left( \phi_j^+(x, y, k) - \frac{\phi_j^+(x, y, k_j^+)}{k - k_j^+} \right) = \beta_j^+ \phi_j^+(x, y, k_j^+),
\]

where

\[
\phi_j^+(x, y, k) = \psi_j^+(x, y, k) e^{i(kx - k^2 y)}, \quad \phi_j^-(x, y, k) = \psi_j^-(x, y, k) e^{i(kx - k^2 y)}.
\]

### 3 Time Dependence

The potential \( u \) can be recovered from the direct scattering problem. After some algebra, it is formulated by\cite{3} (also cf. [14–15])

\[
u(x, y) = 2 \left( \sum_{j=1}^{n_+} \psi_j^+(x, y) + \sum_{j=1}^{n_-} \psi_j^-(x, y) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(k, l) e^{-i(\gamma x + \gamma y, k, l)} \psi_j^-(x, y, l) \, dk \, dl \right),
\]  

where \( f(k, l) \) is a scattering data which will be determined later, \( \psi_j^+(x, y, k) \) and \( \psi^-(x, y, k) \) are governed by the following linear integral equation set:

\[
(x - 2k_j^+ y + \beta_j^+) \psi_j^+(x, y) + \sum_{m=1, m \neq j}^{n_\pm} \frac{i \psi_m^+(x, y)}{k_m^+ - k_j^+} + \sum_{m=1}^{n_\pm} \frac{i \psi_m^-(x, y)}{k_m^+ - k_j^+} = 1 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z, l) e^{-i(\gamma x + \gamma y, z, l)} \psi^-(x, y, l) \, dz \, dl, \quad (j = 1, 2, \ldots, n^\pm),
\]  

\[
\psi^-(x, y, k) + \sum_{m=1}^{n_+} \frac{i \psi_m^+(x, y)}{k_m - k} + \sum_{m=1}^{n_-} \frac{i \psi_m^-(x, y)}{k_m^+ - k} = 1 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z, l) e^{-i(\gamma x + \gamma y, z, l)} \psi^-(x, y, l) \, dz \, dl, \quad \text{Im} \, k = 0,
\]

where one should note that in the integrand of (24b) the Jost function is always \( \psi^-(x, y, l) \), and in (24c) the “real number” \( k \) comes from a limit in the lower half complex plane.

Now what we need is to insert time evolution in the above formula. To do that, we have to determine the time evolution for all the scattering data (20).

For convenience we introduce an operator \( M(k) \) and rewrite the time part (3b) of the Lax pair (3) as

\[
M(k) \phi(t, x, y, k) = 0,
\]

i.e.

\[
M(k) = i \partial_t - \frac{1}{4} y [4 \partial_{\theta x} + 6 u \partial_x + (3 u_x - 3 i \theta^{-1} u_y)] - \frac{1}{2} x (\partial_{\theta x} + 2 u) - \frac{1}{2} \partial_x - \frac{1}{4} (\theta^{-1} u) - \alpha(k).
\]

Noting that the variable \( k \) in the above is only identified by \( \alpha(k) \), we then have a relation

\[
M(k) = M(l) + \alpha(l) - \alpha(k).
\]
Since the solution \( u \) we seek and its \( x \)-derivatives vanish rapidly when \( |x| \) and \( |y| \to \infty \), it then follows from (3b) that for large \( x, y \) we have asymptotics

\[
i \phi_t - iy \phi_{xxx} - \frac{1}{2} x \phi_{xx} - \frac{1}{2} \phi_x - \alpha(k) \phi \sim 0, \quad \text{Im} \, k = 0. \tag{28}
\]

Inserting the asymptotics \( \phi(t, x, y, k) \to e^{i(kx-k^2y)} \), \( |k| \to \infty \) into the above, one immediately gets

\[
k_t = \frac{1}{2} k^2, \quad \alpha(k) = -\frac{1}{2} ik, \quad \text{Im} \, k = 0. \tag{29}
\]

Return to the \( \phi \) version of (16a), i.e.,

\[
\phi^+(t, x, y, k) - \phi^-(t, x, y, k) = \int_{-\infty}^{\infty} f(t, k, l) \phi^-(t, x, y, l) \, dl. \tag{30}
\]

This equation applied by the operator \( M(k) \) with \( \alpha(k) = -ik/2 \) as well as using (25) and (27) yield

\[
0 = M(k) \int_{-\infty}^{\infty} f(t, k, l) \phi^-(t, x, y, l) \, dl = i \int_{-\infty}^{\infty} f_t(t, k, l) \phi^-(t, x, y, l) \, dl + \int_{-\infty}^{\infty} f(t, k, l)(\alpha(l) - \alpha(k)) \phi^-(t, x, y, l) \, dl, \tag{31}
\]

which further gives

\[
f_t(t, k, l) = \frac{1}{2} (l - k) f(t, k, l), \quad \text{Im} \, k = \text{Im} \, l = 0, \tag{32}
\]

(29) and (32) provide time evolution for continuous spectrum.

Next, we search for time evolution of the discrete spectrum. Rewrite (17) in terms of \( \phi^+ \) as

\[
\phi^+(t, x, y, k) = e^{i(kx-k^2y)} + \sum_{j=1}^{n^+} \frac{i \phi^+_{j}(t, x, y, k)}{k - k_j^+} + e^{i(kx-k^2y)} \phi^+(t, x, y, k), \tag{33}
\]

where \( \phi^+(t, x, y, k) \) and \( \phi^+_{j}(t, x, y, k) \) are defined as in (23). Substitute (33) into (28), and the resulting relation allows to be analytically continued to the upper half plane. We rewrite it as

\[
- \sum_{j=1}^{n^+} \frac{i(k^2/2 - k_j^+ t) \phi^+_{j}(t, x, y, k)}{(k - k_j^+)^2} + \sum_{j=1}^{n^+} \frac{i A^+(k) \phi^+_{j}(t, x, y, k)}{k - k_j^+} + B^+(t, x, y, k) \sim 0, \quad \text{Im} \, k \geq 0, \tag{34}
\]

when \( |x|, |y| \) go to \( \infty \), where \( A^+(k) \) is an operator

\[
A^+(k) = i \partial_t - iy \partial_{xxx} - \frac{x}{2} \partial_{xx} - \frac{1}{2} \partial_x + \frac{i}{2} k, \tag{35}
\]

and

\[
B^+(t, x, y, k) = A^+(k)[e^{i(kx-k^2y)}(1 + \theta^+(t, x, y, k))].
\]

Multiplying (34) by \((k - k_m^+)^2\) and letting \( k \to k_m^+ \), from the leading term we find

\[
k_m^{+} = \frac{1}{2} k_m^2. \tag{36}
\]

Substituting the above into (34), when \( k \to k_m \) the leading term in this turn becomes of \( O(1/(k - k_m^+)) \) and its coefficient obeys

\[
A^+(-k^+_m) \phi^+_m(t, x, y, k^+_m) \sim 0, \quad (|x|, |y| \to \infty). \tag{37}
\]

Meanwhile, noting that \( A^+(k) \phi^+(t, x, y, k) \sim 0 \) (i.e. (28) with analytical extension) and

\[
A^+(k) \frac{\phi^+_m(t, x, y, k^+_m)}{k - k_m^+} = A^+(-k^+_m) \frac{\phi^+_m(t, x, y, k^+_m)}{k - k_m^+}, \tag{38}
\]

it then follows from that

\[
A^+(k) \left( \phi^+(t, x, y, k) - \frac{i \phi^+_m(t, x, y, k^+_m)}{k - k_m^+} \right) \sim 0, \quad (|x|, |y| \to \infty), \tag{39}
\]

which, together with (22) and (37), yields the following time evolution for \( \beta^+_m \):

\[
\beta_{m,t}^+ = -k_m^+ \beta_m^+. \tag{40}
\]
After similar discussion for \( \{k_m^-\} \) and \( \{\beta_m\} \), we can collect the time evolution information for all the scattering data, which is concluded as the following.

**Proposition 1** The time dependence of the scattering data (20) is

\[
\begin{align*}
  f(t, k, l) &= f(0, k, l) e^{(l-k)t/2}, \quad k_l = \frac{1}{2} k^2, \quad \text{Im} \, k = \text{Im} \, l = 0; \quad (41a) \\
  k_j^\pm(t) &= \frac{2k_j^\pm(0)}{-tk_j^\pm(0) + 2}, \quad \beta_j^\pm(t) = \frac{b_{j,0}^\pm}{k_j^\pm(0)} \quad j = 1, 2, \ldots, n^\pm, \quad (41b)
\end{align*}
\]

where \( b_{j,0}^\pm \) are complex constants.

### 4 Multi-lump Solutions

We look for a reflectionless real potential \( u \), i.e. \( f(t, k, l) = 0 \) and \( n^- = n^+ = n \), \( k_j^- = k_j^+ \) and \( \beta_j^- = \beta_j^+ \). In this case, \( u \) (24a) and the determining equation set reduce to\(^3\) (also cf.\(^{14-15}\))

\[
u(t, x, y) = 2 \left[ \sum_{j=1}^{n} \left( \psi_j^+(t, x, y) + \psi_j^-(t, x, y) \right) \right], \quad (42)
\]

and

\[
\begin{align*}
  (x - 2k_j^+ y + \beta_j^+) \psi_j^+(t, x, y) + \sum_{m=1, m \neq j}^{n} \frac{i \psi_m^+(t, x, y)}{k_m^+ - k_j^+} + \sum_{m=1}^{n} \frac{i \psi_m^+(t, x, y)}{k_m^+ - k_j^+} &= 1, \quad (43a) \\
  (x - 2k_j^{-} y + \beta_j^-) \psi_j^-(t, x, y) + \sum_{m=1}^{n} \frac{i \psi_m^-(t, x, y)}{k_m^- - k_j^-} + \sum_{m=1, m \neq j}^{n} \frac{i \psi_m^-(t, x, y)}{k_m^- - k_j^-} &= 1, \quad (43b)
\end{align*}
\]

where \( \{k_j^+\} \) and \( \{\beta_j^+\} \) are defined in (41b).

Equation (42) can be written as a simple form\(^{16-17}\)

\[
u(x, y, t) = 2(\ln F)_{xx}, \quad (44)
\]

where \( F \) is the determinant of the coefficient matrix of linear equation set (43), i.e.

\[
F = \text{det} \left[ (x - 2k_j(t)y + \beta_j(t)) \delta_{j,l} - i \frac{1 - \delta_{j,l}}{k_j(t) - k_l(t)} \right]_{2n \times 2n}, \quad (45)
\]

where

\[
k_j(t) = k_j^+(t), \quad k_{n+j}(t) = k_j^-(t), \quad \beta_j(t) = \frac{b_{j,0}^+}{k_j^+}, \quad b_{j,0} = b_{j,0}^+, \quad b_{n+j,0} = b_{j,0}^-, \quad (j = 1, 2, \ldots, n). \quad (46)
\]

The simplest case is of \( n = 1 \)

\[
F = (\xi - 2k_{1R}(t) \eta)^2 + 4k_{1I}(t) \eta^2 + \frac{1}{4k_{1I}(t)^2}, \quad (47)
\]

where

\[
\xi = x - \frac{k_{1R}(t)}{k_{1I}(t)} \beta_{1I}(t) + \beta_{1R}(t), \quad \eta = y - \frac{\beta_{1I}(t)}{2k_{1I}(t)}, \quad k_1(t) = k_{1R}(t) + i k_{1I}(t), \quad \beta_1(t) = \beta_{1R}(t) + i \beta_{1I}(t),
\]

and the explicit forms of the real and imaginary parts of \( k_1(t) \) and \( \beta_1(t) \) are

\[
k_{1R}(t) = \frac{4k_{1R}(0) - 2k_{1R}(0) t - 2k_{1I}(0) t}{(2 - k_{1R}(0)^2 + k_{1I}(0) t)^2}, \quad k_{1I}(t) = \frac{4k_{1I}(0)}{(2 - k_{1R}(0)^2 + k_{1I}(0) t)^2},
\]

\[
\beta_{1R}(t) = \frac{2k_{1R}(t) k_{1I}(t) b_{1,0R} + (k_{1I}(0) t - k_{1I}(t)) b_{1,0R}}{(k_{1I}(t)^2 + k_{1I}(t)^2)^2},
\]

\[
\beta_{1I}(t) = -\frac{2k_{1R}(t) k_{1I}(t) b_{1,0R} + (k_{1R}(t) - k_{1I}(t)) b_{1,0I}}{(k_{1I}(t)^2 + k_{1I}(t)^2)^2},
\]

where \( k_1(0) = k_{1R}(0) + i k_{1I}(0), \quad b_{1,0} = b_{1,0R} + ib_{1,0I}. \) The lump wave corresponding to (47) is then described as

\[
u(t, x, y) = 4 - (\xi - 2k_{1R}(t) \eta)^2 + 4k_{1I}(t) \eta^2 + 1/4k_{1I}(t)^2, \quad (48)
\]

It is not difficult to see that the condition for a non-singular lump solution is \( k_{1I}(t) \neq 0 \). This can be guaranteed by taking \( k_{1I}(0) \neq 0 \) and hereafter we take \( k_{1I}(0) > 0 \) so that \( k_1(t) \) is located in upper half complex plane.
In order to make a comparison with the lump wave in uniform media, we give 1-lump solution of the isospectral KPI equation (1):\[^{[4]}\]

\[\tilde{u}(t, x, y) = 4\frac{-\tilde{\xi}^2 - 2\tilde{k}_{1R}\tilde{\eta}^2 + 4\tilde{k}_{1R}^2\tilde{\eta}^2 + 1/4\tilde{k}_{1I}^2}{\tilde{\xi}^2 - 2\tilde{k}_{1R}\tilde{\eta}^2 + 4\tilde{k}_{1R}^2\tilde{\eta}^2 + 1/4\tilde{k}_{1I}^2} \] (49)

where

\[\tilde{\xi} = x - 12(\tilde{k}_{1R}^2 + \tilde{k}_{1I}^2) t + \tilde{\gamma}_{1R} + \frac{\tilde{k}_{1R}\tilde{\gamma}_{1I}}{\tilde{k}_{1I}}, \quad \tilde{\eta} = y - 12\tilde{k}_{1R} t - \frac{\tilde{\gamma}_{1I}}{2\tilde{k}_{1I}}, \] (50)

in which \(\tilde{k}_{1} = \tilde{k}_{1R} + i\tilde{k}_{1I}\) and \(\tilde{\gamma}_{1} = \tilde{\gamma}_{1R} + i\tilde{\gamma}_{1I}\) are complex parameters. For any given \(t\), (49) is a lump wave of which the vertex is located at

\[ (x, y) = \left( 12(\tilde{k}_{1R}^2 + \tilde{k}_{1I}^2) t + \tilde{\gamma}_{1R} + \frac{\tilde{k}_{1R}\tilde{\gamma}_{1I}}{\tilde{k}_{1I}}, \quad 12\tilde{k}_{1R} t - \frac{\tilde{\gamma}_{1I}}{2\tilde{k}_{1I}} \right). \]

The value of \(\tilde{u}\) at the vertex, i.e., the amplitude is \(16\tilde{k}_{1R}^2\), and the vertex moves with velocity \((\tilde{v}_x, \tilde{v}_y) = (12(\tilde{k}_{1R}^2 + \tilde{k}_{1I}^2), 12\tilde{k}_{1R})\). Obviously, both amplitude and velocity of the lump wave (49) are independent of time. Now let us return to the lump wave (48). Its vertex is located at

\[ (x, y) = \left( \frac{k_{1R}(t)}{k_{1I}(t)} \beta_{11}(t) - \beta_{11}(t), \quad \frac{\beta_{11}(t)}{2k_{1I}(t)} \right), \] (50)

its amplitude is \(16k_{1I}^2(t)\), and the velocity of the wave is \((v_x, v_y)\) where

\[ v_x = 3\frac{(k_{1R}^2(t) - k_{1I}^2(t))b_{1,0I} + 2k_{1R}(t)k_{1I}(t)\beta_{11}b_{1,0R}}{2k_{1I}(t)(k_{1R}^2(t) + k_{1I}^2(t))^2}, \] (51b)

\[ v_y = \frac{-2k_{1R}^3(t)b_{1,0I} + (3k_{1R}^2(t)k_{1I}(t) + k_{1I}^3(t))\beta_{11}b_{1,0R}}{2k_{1I}(t)(k_{1R}^2(t) + k_{1I}^2(t))^2}. \] (51b)

Obviously, both amplitude and velocity are related to time, which acts as characteristics of wave prolongation in non-uniform media.\[^{[6-7]}\]

We depict such a time-dependent lump wave in Fig. 1.

![Fig. 1](image)

**Fig. 1** 1-lump solution given by (44) with (49) for \(k_1(0) = i, b_{1,0} = 1\).

Although each single lump wave possesses time-dependent amplitude and velocity, two lump waves can also exhibit scattering-like interaction. A 2-lump solution is given by (44) where \((n = 2\) in (45))

\[F = \det \begin{pmatrix}
X_1 - iY_1 & -\frac{1}{k_{1I}(t) - k_{2I}(t)} & -\frac{1}{k_{1R}(t) - k_{2R}(t)} & -\frac{1}{k_{1I}(t) - k_{2I}(t)} \\
-\frac{1}{k_{2I}(t) - k_{1I}(t)} & X_2 - iY_2 & -\frac{1}{k_{2R}(t) - k_{1R}(t)} & -\frac{1}{k_{2I}(t) - k_{1I}(t)} \\
-\frac{1}{k_{1I}(t) - k_{2I}(t)} & -\frac{1}{k_{1R}(t) - k_{2R}(t)} & X_1 + iY_1 & -\frac{1}{k_{1I}(t) - k_{2I}(t)} \\
-\frac{1}{k_{2I}(t) - k_{1I}(t)} & -\frac{1}{k_{2R}(t) - k_{1R}(t)} & -\frac{1}{k_{2I}(t) - k_{1I}(t)} & X_2 + iY_2
\end{pmatrix}, \] (52)

with

\[X_1 = x - 2k_{1R}(t)y + \frac{2k_{1I}(t)k_{1R}(t)b_{1,0I} + (k_{1R}(t) - k_{1I}(t))b_{1,0R}}{(k_{1R}(t) + k_{1I}(t))^2}, \]
\[Y_1 = -2k_{1I}(t)y + \frac{-2k_{1R}(t)b_{1,0I} + (3k_{1R}(t)k_{1I}(t) + k_{1I}^3(t))b_{1,0R}}{2k_{1I}(t)(k_{1R}(t) + k_{1I}(t))^2}, \]
\[X_2 = x - 2k_{2R}(t)y + \frac{2k_{2I}(t)k_{2R}(t)b_{2,0I} + (k_{2R}(t) - k_{2I}(t))b_{2,0R}}{(k_{2R}(t) + k_{2I}(t))^2}, \]
\[
Y_2 = -2k_{21}(t)y + \frac{-2k_{3R}^3(t)b_{2,0} + (3k_{2R}^2(t)k_{21}(t) + k_{21}^3(t))b_{2,0}R}{2k_{21}(t)(k_{2R}^2(t) + k_{21}^2(t))},
\]
where \(k_j(t) = k_{jR}(t) + ik_{jI}(t)\), \(b_{j,0} = b_{j,0R} + ib_{j,0I}\), \(j = 1, 2\). The interaction of two lumps is shown in Fig. 2.

![Fig. 2 2-lump solution given by (44) with (52))for \(k_1(0) = i, k_2(0) = 1 + 2i, b_1,0 = b_2,0 = 1\).](image)

5 Conclusions

We have derived \(N\)-lump solutions of the non-isospectral KPI equation through IST. The equation can describe lump waves in non-uniform media. We investigated dynamics of 1-lump solution and show how its amplitude and velocity vary with time. Scattering interactions of two lump waves are also illustrated. The isospectral KPI equation is used to model waves in thin films with high surface tension.\[18\] We have seen that our lump solution (48) is also an algebraically decaying (in terms of \((x, y)\)) localized solutions, which is similar to the isospectral case, but in non-isospectral case the amplitude is time dependent. We hope such a solution can be related to waves in thin films but with non-uniform media effect.

References