Quasideterminant Solutions of a Noncommutative Nonisospectral Kadomtsev–Petviashvili Equation*

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(Received August 26, 2010)

Abstract Solutions of a noncommutative nonisospectral Kadomtsev–Petviashvili equation are given in terms of quasiwronskian and quasigrammian respectively. These solutions are verified by direct substitutions. Dynamics of some obtained solutions are illustrated.

PACS numbers: 02.30.Ik, 05.45.Yv

1 Introduction

The arise of some noncommutative systems is related to the quantization of the phase space. Mathematically, a noncommutative version of a usual (commutative) Lax integrable equation can be derived from the compatibility of the same Lax pair but the potential function and its derivatives are not assumed to be commutative. Solutions of several such noncommutative systems are expressed via quasideterminants,1–2 which provide a value defined on a noncommutative ring. The ways which are shown to be valid for getting quasideterminant solutions include Gelfand–Dikey’s approach,3–4 Darboux transformation,5–6 direct verification,7 and so forth.

In Ref. 7 Gilson and Nimmo expressed the potential and its derivatives in a noncommutative Kadomtsev–Petviashvili (ncKP) equation in terms of special quasideterminants and the direct substitution showed that the equation vanishes. This is something like the Wronskian approach,8 where bilinear equations are reduced to some identities like Plüker relation, but the difference is in noncommutative case nothing remains after substitution.

Many classical solving methods, such as the inverse scattering transform and bilinear method, have been used to find solutions for nonisospectral systems, which are used to describe solitary waves in non-uniform media (cf., for example, Refs. 9–13). In this paper, we would like to apply Gilson and Nimmo’s direct approach to a noncommutative nonisospectral Kadomtsev–Petviashvili (ncKP) equation,

\[
v_{xt} + y[v_{xxxx} + 3v_{yy} + 3(v_x v_{xx} + v_{xx} v_x)] = 0 \tag{1}
\]

where \( v \) and its derivatives belong to some noncommutative ring \( R \), and \( [\cdot, \cdot] \) is defined as \([A, B] = AB - BA, \forall A, B \in R\). This ncKP equation shares the same Lax pair with the commutative non-isospectral KP equation

\[
v_{xt} + y(w_{xxxx} + 3w_{yy} + 6w_{yy} v_{xx}) + 2w_{xy} + 4v_y = 0 \tag{2}
\]

The Lax pair is

\[
L \phi = 0, \quad M \phi = 0,
\]

but here \( \phi, v \) and their derivatives belong to the noncommutative ring \( R \). We note that, actually, both Eqs. (1) and (2) are the potential forms of their counterparts. Replacing \( v_x \) by \( u \) in Eq. (1), the ncKP is

\[
u_x + y[u_{xxxx} + 3u_{yy} + 3(u_{xx} u_{xx} + u_{xx} u_x)] \tag{4}
\]

while in the following we still focus on the potential form (1), and solutions to Eq. (4) can be recovered from

\[
u = \partial_x u \tag{5}
\]

The non-isospectral KP equation (2) can describe line solitons in non-uniformed media,13,15–18 It admits N-soliton solutions in Wronskian form as well as Grammian form, but with different Wronskian condition and Grammian condition (cf. Refs. [13, 15]).

In this paper we will solve the ncKP equation (1) by the direct approach given in Ref. [7]. We will first, in Sec 2, introduce some basic notations and properties of

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*Supported by the National Natural Science Foundation of China under Grant No. 11071157, Shanghai Leading Academic Discipline Project under Grant No. 550101, and Beijing Natural Science Foundation under Grant No. 1101024 and PHR (IHLB)

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quasideterminants. Then in Sec. 3 we present the main results, solutions in quasiwronskian and quasigrammian forms. We will also be interested in the dynamics of the obtained solutions. This will be illustrated in Sec. 4 with an example of a $2 \times 2$ matrix version of non-isospectral KP equation.

2 Quasideterminants

For the convenience and completeness of the paper, let us recall some basic notations and properties of a quasideterminant. These can be found from the review papers\cite{1-2} and many recent papers on solutions to non-commutative systems.

Suppose that $A$ is an $n \times n$ matrix over the noncommutative ring $\mathcal{R}$. Each element $a_{i,j}$ of $A$ can lead to a quasideterminant written as $|A|_{i,j}$, which is also an element of $\mathcal{R}$. $\{|A|_{i,j}\}$ are defined recursively by

$$|A|_{i,j} = a_{i,j} - r^j_i(A^{-1})^{-1}c_j^i, \quad A^{-1} = (|A|_{i,j})_{i,j=1,...,n}. \tag{6}$$

In the above $r^j_i$ represents the $i$-th row of $A$ with the $j$-th element removed, $c_j^i$ represents the $j$-th column with the $i = th$ element removed and $A^{i,j}$ the submatrix obtained by removing the $i$-th row and the $j$-th column from $A$.

Quasideterminants can be also denoted as shown below by boxing the entry about which the expansion is made:

$$|A|_{i,j} = \left| A^{i,j}_{r_j^i} \right|. \tag{7}$$

In the paper we only use the quasideterminants, which are expanded about the last entry of the last column:

$$\begin{vmatrix} A & B \\ C & d \end{vmatrix} = d - CA^{-1}B, \tag{8}$$

where $A$ is an $N \times N$ matrix over $\mathcal{R}$, $C$, and $B$ are row and column vectors over $\mathcal{R}$ and $d \in \mathcal{R}$. We suppose that $A$, $B$, $C$, $d$ are functions of some variable (like $x$, $y$ or $t$) and let prime denote the derivative w.r.t. the variable. Then the derivative of the above quasideterminant can be written as

$$\begin{vmatrix} A & B \\ C & d \end{vmatrix}' = d' - C' A^{-1}B + CA^{-1}A'B - CA^{-1}B'. \tag{9}$$

This can alternatively be expressed as\cite{7}

$$\begin{vmatrix} A & B \\ C & d \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & d' \end{vmatrix} + \sum_{k=1}^N \begin{vmatrix} A & e_k \\ C & 0 \end{vmatrix} \begin{vmatrix} A & B \\ (A_k)^' & (B_k)^' \end{vmatrix}, \tag{10a}$$

or

$$\begin{vmatrix} A & B \\ C & d \end{vmatrix}' = \begin{vmatrix} A & B' \\ C & d' \end{vmatrix} + \sum_{k=1}^N \begin{vmatrix} A & (A_k)' \\ C & (C_k)' \end{vmatrix} \begin{vmatrix} A & B \\ e_k & 0 \end{vmatrix}. \tag{10b}$$

Here $e_k$ is the $N$-th order column vector over $\mathcal{R}$ where only the $k$-th element is 1 and others are 0, (1 and 0 are the unit and zero in $\mathcal{R}$, respectively). $T$ means transposition, $Z^k$ denotes the $k$-th row and $Z_k$ the $k$-th column of a matrix $Z$. Equation (10) will be used to get derivatives for a quasiwronskian.

If the matrix $A$ in Eq. (8) has the grammian-like property that its derivative is a scalar product (cf. Ref. \cite{7})

$$A' = \sum_{i=1}^s E_i F_i^T, \tag{11}$$

where $E_i$ and $F_i$ are $N$-th column vectors over $\mathcal{R}$, then the derivative (9) can be expressed as\cite{7}

$$\begin{vmatrix} A & B \\ C & d \end{vmatrix}' = \begin{vmatrix} A & B \\ C & 0 \end{vmatrix} + \sum_{i=1}^s \begin{vmatrix} A & E_i \\ C & F_i^T \end{vmatrix} \begin{vmatrix} A & B \\ 0 & 0 \end{vmatrix}. \tag{12}$$

This formula will be used to get derivatives for a quasigrammian.

3 Solutions to the ncnKP Equation (1)

3.1 Quasiwronskian Solutions

The quasiwronskian over $\mathcal{R}$ that we use is defined as the following\cite{7}

$$Q(i,j) = \begin{vmatrix} \tilde{\Theta} & e_{N-j} \end{vmatrix}, \tag{13}$$

where

$$\tilde{\Theta} = \hat{\Theta}(x,y,t)$$

is an $N \times N$ Wronskian matrix in terms of the variable $x$, $y$, and $t$ and row vector $\Theta = (\theta_1, \ldots, \theta_N)$, and $\Theta^{(s)}$ stands for the derivative $(\partial / \partial x)^s \Theta$. The complement to this definition is\cite{7}

$$Q(i,j) = \begin{cases} 0, & N - j \notin \{1, 2, \ldots, N\}, \\
-1, & i + j + 1 = 0, \\
0, & i < 0 \text{ or } j < 0 \text{ and } i + j + 1 \neq 0. \tag{14} \end{cases}$$

For the quasiwronskian solutions to the ncnKP equation (1) we have the following theorem.

**Theorem 1** The ncnKP equation (1) has a quasiwronskian solution

$$v = -2Q(0,0), \tag{15}$$

where $Q(i,j)$ is a quasiwronskian defined as in Eq. (13) and the Wronskian vector $\Theta$ satisfies

$$\Theta_y = \Theta_{xx}, \tag{16a}$$

$$\Theta_t = -4y \Theta_{xxx} - 2x \Theta_{xx} + 2(N - 1) \Theta_x. \tag{16b}$$

**Proof** Using the formula (10) and the relation (16a) one can get\cite{7}
\[
\frac{\partial}{\partial x} Q(i, j) = Q(i + 1, j) - Q(i, j + 1) + Q(i, 0)Q(0, j), \\
\frac{\partial}{\partial y} Q(i, j) = Q(i + 2, j) - Q(i, j + 2) + Q(i, 1)Q(0, j) + Q(i, 0)Q(1, j).
\]  \hspace{1cm} (17)

Similarly, from Eq. (16b) we have
\[
\frac{\partial Q(i, j)}{\partial t} = -4y[Q(i + 3, j) - Q(i, j + 3) + Q(i, 2)Q(0, j) + Q(i, 1)Q(1, j) + Q(i, 0)Q(2, j)] - 2x[Q(i + 2, j) - Q(i, j + 2) + Q(i, 1)Q(0, j) + Q(i, 0)Q(1, j)].
\]  \hspace{1cm} (18)

Then using the above formulas and the relation (15) we can express derivatives of \( Q \) w.r.t. \( x \) and \( y \) as are the same as in Ref. [7] while for the completeness of the paper we list them in Appendix. \( v_{xt} \) is different, which is
\[
v_{xt} = 8y[Q(0, 4) - Q(1, 3) - Q(3, 1) + Q(4, 0) - Q(0, 0)Q(0, 3) + Q(3, 0)Q(0, 0) - 2Q(2, 0) - 8Q(0, 2) + 4Q(0, 0)Q(1, 0) + Q(1, 1)] + 4(1, 0)Q(0, 1)Q(0, 0).
\]  \hspace{1cm} (19)

Finally, a direct substitution shows that the left hand side of the ncnKP equation (1) vanishes. Thus we have completed the proof. \( \square \)

3.2 Quasigrammian Solutions

Now we consider the following quasigrammian over \( \mathcal{R} \), defined as\[7\]
\[
R(i, j) = (-1)^i \left[ \begin{array}{c}
\Omega \\
\Phi^{(i)} \\
\end{array} \right],
\]  \hspace{1cm} (20)

where \( \Omega = \Omega(\Phi, \Psi) \) is an \( N \times N \) Grammian matrix in the following form
\[
\Omega(\Phi, \Psi) = \int \Psi \Phi dx + \Delta,
\]  \hspace{1cm} (21)

\( \Phi = (\phi_1, \ldots, \phi_N), \Psi = (\psi_1, \ldots, \psi_N)^T, \Delta = (c_{ij})_{N \times N} \).

\( \phi_j, \psi_j, c_{ij} \in \mathcal{R} \), both \( \phi_j \) and \( \psi_j \) are functions of \((x, y, t)\) while \( c_{ij} \) are constant. \( \Psi^{(j)} \) still stands for the derivative \( (\partial/\partial x)^j \Psi \) and same for \( \Phi^{(j)} \).

Suppose that \( \Phi \) and \( \Psi \) satisfy
\[
\Phi_y = \Phi_{xx}, \quad \Psi_t = -4y\Phi_{xxx} - 2x\Phi_{xx} - 2\Phi_x,
\]
\[
\Psi_y = -\Psi_{xx}, \quad \Psi_t = -4y\Psi_{xxx} + 2x\Psi_{xx} - 2\Psi_x.
\]  \hspace{1cm} (22)

These conditions yield
\[
\Omega(\Phi, \Psi)_x = \Psi \Phi, \quad \Omega(\Phi, \Psi)_y = \Psi_x \Phi - \Psi \Psi_x,
\]
\[
\Omega(\Phi, \Psi)_t = -4y(\Phi_{xxx} - \Psi_x \Phi_x) + 2x(\Psi_{xx} \Phi - \Psi_x \Psi_x).
\]  \hspace{1cm} (23)

Then, similar to Ref. [6] we have
\[
\frac{\partial}{\partial x} R(i, j) = R(i + 1, j) - R(i, j + 1) + R(i, 0)R(0, j),
\]
\[
\frac{\partial}{\partial y} R(i, j) = R(i + 2, j) - R(i, j + 2) + R(i, 1)R(0, j)
\]
\[
+ R(i, 0)R(1, j), \frac{\partial}{\partial t} R(i, j)
\]
\[ v_{11,xt} = -4v_{11y} + v_{21}v_{12x} - v_{12}v_{21x} - 2xv_{11xy} - y[3v_{11yy} - 3v_{21y}v_{12x} + 3v_{12y}v_{21x} + 3v_{212x}v_{v11} + v_{112xx}]x, \]
\[ v_{12,xt} = -4v_{12y} - v_{11}v_{12x} + v_{22}v_{12x} - v_{12}v_{22x} + v_{11x}x - 2xv_{11xy} - y[3v_{12yy} + 3v_{11y}v_{12x} - 3v_{22y}v_{12x} - 3v_{12y}v_{12x} + 3v_{12x}(v_{11x} + v_{22x}) + 3v_{12xxx}]x, \]
\[ v_{21,xt} = -4v_{21y} + v_{11}v_{21x} - v_{22}v_{21x} - v_{21}v_{21x} + v_{22x} - 2xv_{21xy} - y[3v_{21yy} - 3v_{11y}v_{21x} + 3v_{22y}v_{12x} + 3v_{12y}v_{12x} + 3v_{21x}(v_{11x} + v_{22x}) + 3v_{21xxx}]x, \]
\[ v_{22,xt} = -4v_{22y} - v_{21}v_{21x} + v_{21}v_{22x} - 2xv_{22xy} - y[3v_{22yy} + 3v_{11y}v_{12x} - 3v_{12y}v_{21x} + 3v_{212x}v_{v11} + v_{112xx}]x. \] (27)

In the following we present and depict solutions of this $2 \times 2$ matrix ncnKP equation.

### 4.1 Quasiwronskian Case

For the ncnKP equation (1) or (4) with $2 \times 2$ matrix (26), the quasiwronskian solution is

\[
u = u_x, \quad v = -2Q(0,0) = \begin{bmatrix} \tilde{\Theta} \Theta^{(N)} \end{bmatrix} \begin{bmatrix} e_x \\ 0 \end{bmatrix}, \] (28)

To agree with the quasiwronskian condition (16) we take

\[
\theta_j = \begin{bmatrix} \theta_{11}^{(j)} \\ \theta_{12}^{(j)} \\ \theta_{21}^{(j)} \\ \theta_{22}^{(j)} \end{bmatrix}, \quad j = 1, 2, \ldots, N, \] (29a)

where

\[
\theta_{ij}^{(j)} = h_{ij}^{[j]} \omega_{ij}^{[j]}(t) e^{\xi_{ij}^{[j]}} + i j^{[j]} \rho_{ij}^{[j]}(t) e^{-\eta_{ij}^{[j]}}, \] (29b)

\[\ h_{ij}^{[j]}, \ i j^{[j]} \in \mathbb{R} \] and \( \xi_{ij}^{[j]} = \eta_{ij}^{[j]}(t) x + (k_{ij}^{[j]}(t))^{2} y, \)
\[k_{ij}^{[j]}(t) = \frac{1}{2t + e_{ij}^{[j]}}, \quad \omega_{ij}^{[j]}(t) = (k_{ij}^{[j]}(t))^{-1}, \] (29c)

\[
\eta_{ij}^{[j]} = q_{ij}^{[j]}(t) x - (q_{ij}^{[j]}(t))^{2} y, \]
\[
q_{ij}^{[j]}(t) = \frac{1}{-2t + d_{ij}^{[j]}}, \quad \rho_{ij}^{[j]}(t) = (q_{ij}^{[j]}(t))^{-1}, \] (29d)

with arbitrary real parameters \( c_{ij}^{[j]} \) and \( d_{ij}^{[j]} \). We note that \( \theta_{ij}^{(j)} \) can be a summation of more terms so long as each term agrees with Eq. (16).

In the simplest case, i.e., \( N = 1 \), we have a one-soliton solution

\[
u = -2Q(0,0) = -2 \begin{bmatrix} \theta_1^{(1)} 1 \\ 0 \end{bmatrix}, \] (30)

where explicit forms of each \( u_{ij} \) are listed out in Appendix B.

For the ncnKP equation (4), solution is obtained through

\[ u_{ij} = \partial_x u_{ij}, \] (31)

We depict this solution in Fig. 1.

**Fig. 1** One-soliton given by Eq. (31) for \( t = 2, \ c_{11} = 5, \ c_{12} = -5, \ c_{21} = -3, \ c_{22} = 3, \ d_{11} = 5, \ d_{12} = -5, \ d_{21} = -3, \ d_{22} = 3, \ h_{11} = -1, \ h_{12} = -3, \ h_{21} = -3, \ h_{22} = -1, \ l_{11} = 3, \ l_{12} = -1, \ l_{21} = -3, \ l_{22} = 1.**
4.2 Quasigrammian Case

Solutions in quasigrammian case can be given by
\[ u = v_x, \quad v = -2R(0, 0) \begin{bmatrix} \Omega & \Psi \\ \Phi & 0 \end{bmatrix}, \] (32)

For the entries corresponding the 2 × 2 ncnKP equation we take
\[ \phi_j = \begin{pmatrix} h_{11}^{(j)} k_{11}^{(j)}(t) e^{-\gamma_{11}^{(j)}} & h_{12}^{(j)} k_{12}^{(j)}(t) e^{-\gamma_{12}^{(j)}} \\ h_{21}^{(j)} k_{21}^{(j)}(t) e^{-\gamma_{21}^{(j)}} & h_{22}^{(j)} k_{22}^{(j)}(t) e^{-\gamma_{22}^{(j)}} \end{pmatrix}, \]
\[ \psi_j = \begin{pmatrix} i^{(j)} p_{11}^{(j)}(t) e^{-\gamma_{11}^{(j)}} & i^{(j)} p_{12}^{(j)}(t) e^{-\gamma_{12}^{(j)}} \\ i^{(j)} p_{21}^{(j)}(t) e^{-\gamma_{21}^{(j)}} & i^{(j)} p_{22}^{(j)}(t) e^{-\gamma_{22}^{(j)}} \end{pmatrix}, \] (33)

for \( j = 1, 2, \ldots, N \), where \( h_{is}^{(j)}, i^{(j)} \in \mathbb{R} \), \( \zeta_{is}^{(j)} \), and \( \gamma_{is}^{(j)} \) are
\[ \zeta_{is}^{(j)} = k_{is}^{(j)}(t)x + (k_{is}^{(j)}(t))^{2}y, \quad k_{is}^{(j)}(t) = \frac{1}{2t + \zeta_{is}^{(j)}}, \]

\[ \gamma_{is}^{[j]} = p_{is}^{[j]}(t)x - (p_{is}^{[j]}(t))^{2}y, \quad p_{is}^{[j]}(t) = \frac{1}{2t + d_{is}^{[j]}}, \] (34)

with arbitrary real parameters \( c_{is}^{[j]} \) and \( d_{is}^{[j]} \).

In the case of \( N = 1 \), we obtain a one-soliton quasigrammian solution expressed through
\[ v = -2R(0, 0) = -2 \begin{vmatrix} \Omega(\phi_1, \psi_1) & \psi_T \\ \phi_1 & 0 \end{vmatrix}, \] (35)

where
\[ \Omega(\phi_1, \psi_1) = \int \psi_T \phi_1 dx + I_2, \] (36)

and we have taken \( \Delta \) to be the 2 by 2 unit matrix \( I_2 \). Explicit forms of each \( v_{ij} \) are listed out in Appendix C and solutions
\[ u_{ij} = \partial_x v_{ij} \] (37)

are depicted in Fig. 2.

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5 Conclusions

From literatures, we see that quasideterminant has been shown to be a powerful tool to express solutions for noncommutative versions of some well-known integrable systems. In this paper, we have applied quasideterminant to study a noncommutative non-isospectral KP equation. We obtained solutions to the ncnKP in terms of quasiwronskian and quasigrammian respectively and verified by direct substitutions. As a special case, we studied the 2 × 2 matrix non-isospectral KP equation and obtained its explicit solutions from both quasiwronskian solutions and quasigrammian solutions. We note that with some special choices these two kinds of solutions for the system (27) can provide the same solutions, as in usual commutative case.

As for dynamics, we illustrated some obtained explicit solutions of the 2 × 2 matrix non-isospectral KP equation. From Fig. 1 we can observe some “strange” dynamics which are different from the usual. However, this might be typical for coupled systems (cf. Ref. [20] and the dynamics of the Hirota-Satsuma equation (a coupled KdV equation)). For example, in Fig. 1(a) there is a shift in the line soliton \( u_{11} \). This shift might be resulted from \( u_{22} \) depicted in Fig. 1(d). \( u_{12} \) and \( u_{21} \) should, somehow, be related but in this case, their amplitudes are both negative while amplitudes of \( u_{11} \) and \( u_{22} \) are positive. Figure 2 also exhibits interesting behaviors: waves with positive and negative amplitudes interact in each solution. We would like to investigate such interesting dynamics by means of asymptotic analysis in the future.
Appendix A: Derivatives of \( Q(0,0) \)
Here is a list of derivatives of the quasiwronskian \( Q(0,0) \). They are listed out in Ref. [7] except \( v_{x1} \) Eq. (19).
\[
\begin{align*}
    v_x &= -2[Q(1,0) - Q(0,1) + Q(0,0)^2], \\
    v_y &= -2[Q(2,0) - Q(0,2) + Q(0,0)Q(1,0) + Q(1,0)Q(0,0)], \\
    v_{xx} &= -2[Q(2,0) - 2Q(1,1) + Q(0,2) + Q(0,0)(Q(1,0) - 2Q(0,1)) + (2Q(1,0) - Q(0,1))Q(0,0) + 2Q(0,0)^3], \\
    v_{yy} &= -2[Q(0,4) - 2Q(2,2) + Q(4,0) + Q(0,1)(Q(2,0) - 2Q(0,2)) + (2Q(2,0) - Q(0,2))Q(1,0) + Q(0,0)Q(3,0) - 2Q(1,2) + 2Q(1,0)Q(1,0) + (2Q(2,1) - Q(3,0) + 2Q(0,0)Q(1,1))Q(0,0)], \\
    v_{xy} &= -2[Q(1,0) - Q(1,2) - Q(2,1) + Q(0,0)(Q(2,0) - Q(0,2) - Q(1,1) + Q(0,0)Q(1,0) + Q(0,0)Q(0,0)] + (Q(2,0) - Q(0,2) + Q(1,1) + Q(0,0)Q(0,0))Q(0,0) + (Q(1,0)^2 - Q(0,1)^2).
\end{align*}
\]

Appendix B: Explicit Forms of \( v_{ij} \) in Eq. (30)
We omit the superscript “\([1]\)” for simplicity.
\[
\begin{align*}
v_{11} &= 2 \left[ \left( h_{22} e^{\xi_{22}} + l_{22} e^{\eta_{22}} \right) \left( h_{11} e^{\xi_{11}} \right) \left( h_{12} e^{\xi_{12}} \right) \right] - \left( \frac{h_{22} e^{\xi_{22}}}{c_{12} + 2t} - \frac{h_{21} e^{-\eta_{12}}}{c_{12} - 2t} \right), \\
v_{12} &= 2 \left[ \left( h_{22} e^{\xi_{22}} - l_{22} e^{-\eta_{22}} \right) \left( h_{11} e^{\xi_{11}} \right) \left( h_{12} e^{\xi_{12}} \right) \right] - \left( \frac{h_{22} e^{\xi_{22}}}{c_{12} + 2t} - \frac{h_{21} e^{-\eta_{12}}}{c_{12} - 2t} \right), \\
v_{21} &= 2 \left[ \left( h_{22} e^{\xi_{22}} + l_{22} e^{\eta_{22}} \right) \left( h_{11} e^{\xi_{11}} \right) \left( h_{12} e^{\xi_{12}} \right) \right] - \left( \frac{h_{22} e^{\xi_{22}}}{c_{12} + 2t} - \frac{h_{21} e^{-\eta_{12}}}{c_{12} - 2t} \right), \\
v_{22} &= 2 \left[ \left( h_{22} e^{\xi_{22}} - l_{22} e^{-\eta_{22}} \right) \left( h_{11} e^{\xi_{11}} \right) \left( h_{12} e^{\xi_{12}} \right) \right] - \left( \frac{h_{22} e^{\xi_{22}}}{c_{12} + 2t} - \frac{h_{21} e^{-\eta_{12}}}{c_{12} - 2t} \right).
\end{align*}
\]

Appendix C: Explicit Forms of \( v_{ij} \) in Eq. (35)
We omit the superscript “\([1]\)” for simplicity.
\[
\begin{align*}
v_{ij} &= B_{ij} B, \\
B_{11} &= \frac{2h_{11} l_{11} e^{\eta_{11} - \xi_{11}}}{(2t + c_{11})(2t + d_{11})} + \frac{2h_{11} h_{12} l_{12} l_{11} e^{-\gamma_{11} - \gamma_{12} + \eta_{11} + \eta_{12}}}{(2t + c_{11})(2t + d_{11})(c_{11} - d_{11})}, \\
&\quad - \frac{2h_{12} h_{22} l_{22} l_{11} e^{-\gamma_{11} - \gamma_{12} + \eta_{11} + \eta_{22}}}{(2t + c_{11})(2t + d_{11})(c_{22} - d_{22})}, \\
&\quad - \frac{2h_{11} h_{22} l_{22} l_{11} e^{-\gamma_{11} - \gamma_{12} + \eta_{11} + \eta_{12}}}{(2t + c_{11})(c_{11} - d_{11})(2t + d_{11})}, \\
&\quad + \frac{2h_{11} h_{22} l_{22} l_{11} e^{-\gamma_{11} - \gamma_{12} + \eta_{11} + \eta_{22}}}{(2t + c_{11})(2t + d_{11})(c_{22} - d_{22})}, \\
B_{12} &= -\frac{2h_{11} l_{21} e^{\eta_{11} - \xi_{21}}}{(2t + c_{11})(2t + d_{21})} + \frac{2h_{12} l_{12} l_{21} e^{-\gamma_{12} - \gamma_{11} + \eta_{11} + \eta_{12}}}{(2t + c_{11})(c_{11} - d_{12})(2t + d_{12})}, \\
&\quad - \frac{2h_{12} h_{22} l_{22} l_{21} e^{-\gamma_{12} - \gamma_{11} + \eta_{11} + \eta_{22}}}{(2t + c_{11})(c_{22} - d_{21})(2t + d_{22})}, \\
&\quad + \frac{2h_{12} h_{22} l_{22} l_{21} e^{-\gamma_{12} - \gamma_{11} + \eta_{11} + \eta_{22}}}{(2t + c_{11})(2t + d_{21})(c_{21} - d_{22})}, \\
&\quad + \frac{2h_{11} h_{22} l_{22} l_{21} e^{-\gamma_{12} - \gamma_{11} + \eta_{11} + \eta_{22}}}{(2t + c_{11})(c_{11} - d_{12})(2t + d_{21})}, \\
B_{21} &= -\frac{2h_{21} l_{11} e^{\eta_{21} - \xi_{11}}}{(2t + c_{21})(2t + d_{11})} - \frac{2h_{12} h_{21} l_{21} l_{11} e^{-\gamma_{11} - \gamma_{12} + \eta_{11} + \eta_{22}}}{(2t + c_{21})(2t + d_{11})(c_{12} - d_{12})}, \\
&\quad - \frac{2h_{21} h_{22} l_{22} l_{11} e^{-\gamma_{11} - \gamma_{12} + \eta_{11} + \eta_{22}}}{(2t + c_{21})(c_{22} - d_{22})(2t + d_{22})}, \\
&\quad + \frac{2h_{21} h_{22} l_{22} l_{11} e^{-\gamma_{11} - \gamma_{12} + \eta_{11} + \eta_{22}}}{(2t + c_{21})(2t + d_{11})(c_{22} - d_{22})}. \\
\end{align*}
\]
\[
B_{22} = -\frac{2h_{11}h_{22}l_{12}l_{11}e^{-\gamma_{11}l_{12} + \eta_{11}l_{12} + \eta_{22}l_{11}}}{(2t + c_{22})(2t + d_{11})}(c_{11} - d_{11}) + \frac{2h_{11}h_{22}l_{12}l_{11}e^{-\gamma_{11}l_{12} + \eta_{11}l_{12} + \eta_{22}l_{11}}}{(2t + c_{22})(2t + d_{11})}(c_{11} - d_{11})(2t + d_{11}) + \frac{2h_{11}h_{22}l_{12}l_{11}e^{-\gamma_{11}l_{12} + \eta_{11}l_{12} + \eta_{22}l_{11}}}{(2t + c_{22})(2t + d_{11})}(c_{11} - d_{11})(2t + d_{11}) \]

\[
B = 1 + \frac{h_{11}h_{12}l_{12}l_{11}e^{-\gamma_{11}l_{12} + \eta_{11}l_{12} + \eta_{12}l_{12}}}{(c_{12} - d_{12})(c_{11} - d_{11})} + \frac{h_{11}h_{12}l_{12}l_{11}e^{-\gamma_{11}l_{12} + \eta_{11}l_{12} + \eta_{12}l_{12}}}{(c_{12} - d_{12})(c_{11} - d_{11})}(c_{11} - d_{11})(2t + d_{11}) \]

References