Non-isospectral flows of noncommutative differential-difference KP equation

Lin Huang b,c, R. Ilangovane a, K.M. Tamizhmani a,*, Da-jun Zhang b

a Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Puducherry 605 014, India
b Department of Mathematics, Shanghai University, Shanghai 200444, PR China
c School of Mathematical Sciences, Institute of Mathematics and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, PR China

ABSTRACT

We present master symmetries of noncommutative differential-difference KP equation by considering Sato approach, where the field variables are defined over associative algebras. The Lie algebraic structures of generalized and master symmetries are given. They form a Virasoro Lie algebraic structure.

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1. Introduction

Integrable systems are known for its rich and diverse mathematical structures and applications. In the past four decades, the research in searching soliton systems and investigating their integrability properties have made significant progress [1]. Date et al. systematically derived soliton system and their Lax pairs for both continuous and discrete equations by using group theoretic approach in a series of papers [2–6]. It is well-known that Sato theory is a powerful mathematical approach to derive Lax pairs, soliton solutions, conservation laws etc. [7] for KP hierarchy. This approach has further extended to differential difference set up by [8–11] and obtained differential-difference KP equation, generalized symmetries and conservation laws. Zhang et al. continued this analysis and found the conservation laws and solutions of differential-difference KP in a much simpler way [12,13]. Searching for various symmetries, master symmetries and Lie algebraic structures for continuous and lattice equations is an important topic for many years [15–21]. They always play a fundamental role in integrable systems. In [19], Ma proposed a method to construct Lax representation for isospectral and non-isospectral hierarchies of (1 + 1) dimensional evolution equations. Further developments have been made in [16,20,21] for lattice equations as well. Moreover, using this one can study the underlying infinite dimensional Lie algebraic structures of the concerned system. Recently, many studies have been carried out in finding the symmetries and their Lie algebraic structures for differential-difference systems [8,10,12–14]. The non-isospectral flows assume its importance in their own right. It requires a separate study to find out various properties of them concerning the integrability. It is interesting to note that solutions of certain non-isospectral equations were carried out by Zhang et al. [22–24]. In recent years, noncommutative version of soliton equations attracted lot of attention in the area of integrable nonlinear partial differential equations [25–43]. They arise in many situations, where the field variables, for example, might be square matrices or quarternions etc. There is an another way in defining these equations through * or Moyal product [36–41]. In this paper, the non-commutative version is obtained by assuming that the coefficients in the pseudo-difference operator do not commute with each other. In the literature not enough results are available for non-commutative version of soliton equations [28,29,38,41,40]. Hence, there are huge possibilities exist to explore in this domain and obtain interesting results. Olver and Sokolov [26] developed a systematic procedure to find generalized symmetries, recursion operators and Hamiltonian structures of certain (1 + 1) dimensional
noncommutative evolution equations. In a series of papers, Hamanaka extended Sato theory for noncommutative settings and obtained continued noncommutative KP (NCKP) equation [36–38] and its various reductions. A detailed investigation about extended KP and its noncommutative version have been discussed in [43,44]. However, not enough is known about various integrability properties of noncommutative differential-difference soliton equations. In this paper, we follow Lax triad approach. First, we consider isospectral and non-isospectral flows of NCD KP with eigenvalue \( \eta \) and eigenfunction \( \phi \) as follows:

\[
L\phi = \eta \phi, \quad (2.2a)
\]

\[
\phi_y = A_1 \phi, \quad (2.2b)
\]

\[
\phi_t = A_2 \phi, \quad (2.2c)
\]

where \( \eta \lvert_{t=\eta} = 0 \) and \( A_i = (L^i) \), projection into the positive powers of \( \Delta \), satisfying the boundary condition

\[
A_i \lvert_{t=0} = \Lambda^i. \quad (2.3)
\]

Then the compatibility conditions of the linear problems \((2.2)\) lead to

\[
L_y = [A_1, L], \quad (2.4a)
\]

\[
L_t = [A_2, L], \quad (2.4b)
\]

\[
A_{1,t} = A_{2,y} - [A_1, A_2], \quad (2.4c)
\]

From the fact, \( A_i = (L^i) \), all the \( A_i \) can be determined and the first few explicit forms of \( A_i \) are given below:

\[
A_1 = \Delta + u_0, \quad (2.5a)
\]

\[
A_2 = \Delta^2 + (\Delta u_0 + 2u_0)\Delta + (\Delta u_0 + u_0^2 + \Delta u_1 + 2u_1), \quad (2.5b)
\]

\[
A_3 = \Delta^3 + a_1 \Delta^2 + a_2 \Delta + a_3, \quad (2.5c)
\]

with

\[
a_1 = \Delta^2 u_0 + 3\Delta u_0 + 3u_0, \quad a_2 = 2\Delta^2 u_0 + 3\Delta u_0 + 3u_0^2 + 2u_0\Delta u_0 + \Delta u_0 u_0 + (\Delta u_0)^2 + 3u_1 + 3\Delta u_1 + \Delta^2 u_1, \quad a_3 = \Delta^3 u_0 + 3u_0 u_1 + 2u_1 u_0 + 2u_0 \Delta u_0 + (\Delta u_0) u_0 + (\Delta u_0)^2 + (\Delta u_0) \Delta u_1 + 2u_0 \Delta u_1 + (\Delta u_1) u_0 + (\Delta u_0) u_1 + u_1 \Delta^{-1} u_0 + 2\Delta^2 u_1 + 3\Delta u_1 + 3\Delta u_2 + 3u_2 + \Delta^2 u_2 + u_0^3.
\]

Using \((2.5a)\) in \((2.4a)\), we get

\[
u_{0,y} = q_{10} = \Delta u_1, \quad (2.7a)
\]

\[
u_{1,y} = q_{11} = \Delta u_1 + \Delta u_2 + u_0 u_1 - u_1 \Delta^{-1} u_0, \quad (2.7b)
\]

\[
u_{2,y} = q_{12} = \Delta u_3 + \Delta u_2 + u_0 u_2 + u_1 \Delta^{-1} u_1 - u_2 \Delta^{-2} u_0 - u_1 \Delta^{-2} u_0, \quad (2.7c)
\]

From \((2.7)\), we can easily express \( u_j > 0 \) through \( u_0 \), i.e.,

\[
u_1 = \Delta^{-1} u_0 y, \quad (2.8a)
\]

\[
u_2 = \Delta^{-2} u_{0,y} y - \Delta^{-1} u_{0,y} y - \Delta^{-1} (u_0 \Delta^{-1} u_{0,y}), \quad (2.8b)
\]

By employing \((2.4c)\), one can have the evolution equations for \( u_0 \):

\[
u_{0,y} = q_{10} = u_0 y, \quad (2.9a)
\]

\[
u_{0,y} = q_{20} = \Delta u_{0,y} + u_0 u_0 y + u_0 u_0 y + \Delta u_{1,y} + 2u_{1,y} + [\Delta u_1, u_0] + 2[u_1, u_0] - \Delta^2 u_1 - 2\Delta u_1, \quad (2.9b)
\]
\[ u_{0t_1} = \alpha_3 y + \Delta^2 u_0 + \alpha_2 \Delta^2 u_0 + \alpha_2 \Delta u_0 + [\alpha_3, u_0] \]
\[ - \Delta u_1 \]  
(2.9c)

\[ \ldots \]  

After replacing \( u_0(j > 0) \) by \( u_0 \) from (2.8) and \( u_0 \) by \( u \), we have
\[ u_{1t} = K_1 = u_y, \]  
(2.10a)

\[ u_{12} = K_2 = (1 + 2 \Delta^{-1}) u_{yy} - 2u_y + 2u_y u + 2[\Delta^{-1} u_y, u], \]  
(2.10b)

\[ u_{1t} = K_3 = (3 \Delta^{-2} + 3 \Delta^{-1} + 1) u_{yyy} + 3 \Delta^{-1} (u_y^2) + 3 \Delta^{-1} (u_y u_y) u \]
\[ - 6 \Delta^{-1} u_{yy} + 3u_y + 3 \Delta^{-1} (u_y u_y) u + 3 \Delta^{-1} (u_y u_y) u + 3 \Delta^{-1} u_{yy} u \]
\[ - 3u_y + 3u_y^2 + 3u_y u - 6u_y + 2 \Delta u_y + 2[\Delta u_y, u] \]
\[ + [\Delta^{-2} + \Delta^{-1} u_{yy}, u] + [\Delta^{-1} u_y E^{-1} u, u] + 3 \Delta^{-1} u_{yy}, u] + 6u(\Delta^{-1} u_{yy} + 3 \Delta^{-1} u_{yy} u + 3 \Delta^{-1} (u_y u_y) u] \]
\[ + 3 \Delta^{-1} (\Delta^{-1} u_y E^{-1} u_y) u_y + 3 [u, \Delta^{-1} (u_y u_y) u_y], \]  
(2.10c)

\[ \ldots \]  

The above set of equations are the isospectral NCD\( \Delta \)KP hierarchy and observe that Eq. (2.10b) is the noncommutative \( \Delta \)KP equation.

2.2. Non-isospectral NCD\( \Delta \)KP

Setting the spectral parameter as the function of time variable [12]
\[ \eta_t = \eta^r + \eta^{-r} \]
and consider the spectral problem associated to the non-isospectral flows of NCD\( \Delta \)KP equation as
\[ L \phi = \eta \phi, \]  
(2.11a)
\[ \phi_y = A_1 \phi, \]  
(2.11b)
\[ \phi_{tt} = B_r \phi, \]  
(2.11c)

where \( A_1 \) is as defined in (2.5a), and
\[ B_r = b_0 \Delta^r + b_1 \Delta^{r-1} + \cdots + b_r, \quad (r > 0) \]  
(2.12)

is to be determined with the boundary condition
\[ B_r|_{u=0} = y \Delta^r + (y + n) \Delta^{r-1}. \]  
(2.13)

Then the compatibility condition on (2.11) turns out to be
\[ L_y = [A_1, L], \]  
(2.14a)
\[ L_{yy} = [B_r, L] + L^r + L^{r-1}, \]  
(2.14b)
\[ A_1|_{x=0} = B_r|_{x=0} = [A_1, B_r]. \]  
(2.14c)

Using the various values of \( r, r = 1, 2, \ldots \) in (2.14b) along with (2.13), we derive
\[ B_1 = y A_1 + y + n, \]  
(2.15a)
\[ B_2 = y A_2 + y A_1 + \Delta + nu_0 + \Delta^{-1} u_0, \]  
(2.15b)
\[ B_3 = y A_3 + y A_2 + \Delta^2 + (2 nu_0 + \Delta u_0 + \Delta^{-1} u_0) \Delta + \Delta u_0 \]
\[ + \Delta u_0 u_0 + 2 mu_1 + nu_0^2 + u_0 \Delta^{-1} u_0 + \Delta^{-1} (2 u_1 + u_2). \]  
(2.15c)

From (2.14c), we have
\[ u_{0t_1} = y q_{10} + u_0, \]  
(2.16a)

\[ u_{0t_1} = y q_{20} + (y + n) q_{10} + u_0 - u_0 + 2 \Delta u_1 + 2 u_1 \]
\[ + \Delta^{-1} u_0 + [\Delta^{-1} u_0, u_0], \]  
(2.16b)

\[ u_{0t_1} = y q_{20} + (y + n) q_{20} - \Delta^2 u_0 - \Delta (u_0^2) \]
\[ - \Delta^2 u_1 - 3 \Delta u_1 - \Delta u_0 - 3 u_0 \]
\[ - 4 u_1 + u_0 [\Delta^{-1} u_0, u_0] - \Delta u_0 u_0 \]
\[ + [\Delta^{-1} u_0, u_0] + 2[\Delta^{-1} u_1, u_0] \]
\[ + [\Delta^{-1} u_0, u_0] + u_0 + [\Delta^{-1} u_1, u_0] + \alpha_3 \]
\[ + \Delta u_0 + u_0^2 + \Delta u_1 + 2 u_1 + u_0 \Delta^{-1} u_0 \]
\[ + u_0 \Delta^{-1} u_0 + \Delta^{-1} u_1 + u_0 \Delta^{-1} (u_0 u_0), \]
\[ + \Delta^{-1} (u_0 u_0), \]  
(2.16c)

\[ \ldots \]  

Here, \( \alpha_3 \) and \( q_0 \) are described by (2.5) and (2.9). Now, by using the relations (2.8) in (2.16a), (2.16b), (2.16c), . . ., we have \((u_0 = u)\)
\[ u_{1t} = \sigma_1 = y K_1 + u, \]  
(2.17a)
\[ u_{1r} = \sigma_2 = y K_2 + (y + n) K_1 + u_y + \Delta^{-1} u_y + u^2 - u \]
\[ + [\Delta^{-1} u_y, u], \]  
(2.17b)
\[ u_{1s} = \sigma_3 = y K_3 + (y + n) K_2 + [u, \Delta u] - 2 u_y - 2 u^2 - 6 [u, \Delta u] \]
\[ + [u, \Delta u] - 2 [\Delta^2 u, u] + 2 [\Delta^{-1} u, u] + [u, \Delta^{-1} u] \]
\[ + [u, \Delta^{-1} u, u] + [u, \Delta^{-1} u, u] + 5 [\Delta^{-2} u, u] \]
\[ + \Delta^{-1} (u, u) + [\Delta^{-1} u, u] + 3 \Delta^{-1} u_y + u_y + 2 u_y \]
\[ + 3 [\Delta^{-1} u_y, u_y] + 3 \Delta^{-1} u_y E^{-1} u - 3 \Delta^{-1} (u \Delta^{-1} u_y) \]
\[ + 3 \Delta^{-1} (\Delta^{-1} u_y E^{-1} u) + u_{yy} + u_y^3, \]  
(2.17c)

\[ \ldots \]  

The above set of equations are the non-isospectral flows of NCD\( \Delta \)KP equation. Moreover, these flows can be expressed from the following zero-curvature equations as follows:

Isospectral case:
\[ K_1 = A_{1y} = [A_1, A_1], \]  
(2.18a)
\[ A_1|_{u=0} = \Delta^1 \]  
(2.18b)

and non-isospectral case:
\[ \sigma_r = B_{r-1} - [A_1, B_r], \]  
(2.19a)
\[ B_r|_{u=0} = y \Delta^r + (y + n) \Delta^{r-1}. \]  
(2.19b)

3. Lie algebraic structure of the NCD\( \Delta \)KP equation

Let \( \mathcal{A} \) be an associative algebra and \( u \in \mathcal{A} \) and \( D[n, t, u(n, t)] \) be a differential difference polynomial not necessarily commutative. The \( \mathcal{A} \) valued evolution equation is of the form
\[ u_t = K[u], \quad K \in D. \]  
(3.1)

The Gateaux derivative of \( f(u) \in D \) in direction \( h \in D \) with respect to \( u \) is defined as
\[ f'[h] = \frac{d}{dt} f(u + \varepsilon h)|_{\varepsilon \to 0} \]  
(3.2)
from which $D$ forms a Lie algebra according to the following Gateaux commutator:

$$[f,g] = f'[g] - g'[f],$$

(3.3)

where $f,g \in D$. For a differential-difference operator

$$P(u) = \sum_{j=0}^{L} p_j(u)\Delta^j$$

(3.4)

and its Gateaux derivative in the direction $h$ with respect to $u$ is defined by

$$P'[h] = \sum_{j=0}^{L} p'_j[h]\Delta^j.$$  

(3.5)

In addition, it is easy to prove the following lemma.

**Lemma 1.** For the difference operator $B$ in the form (2.12) and $X \in D$, the equation

$$X = B_s - [A_s, B_s], \quad B_{s=0} = 0,$$

(3.6)

only admits zero solution $X = 0, B = 0$.

**Theorem 1.** If we define the following kind of products with operators

$$(A_s, A_t) = A'_s[A_t] - A'_t[A_s] + [A_s, A_t],$$

(3.7a)

$$(A_s, B_t) = A'_s[B_t] - B'_t[A_s] + [A_s, B_t],$$

(3.7b)

$$(B_s, B_t) = B'_s[B_t] - B'_t[B_s] + [B_s, B_t],$$

(3.7c)

then we have

$$[K_s, K_t] = (A_s, A_t)_{y} = -[A_s, (A_s, A_t)],$$

(3.8a)

$$[K_s, \sigma_r] = (A_s, B_t)_{y} = -[A_s, (A_s, B_t)],$$

(3.8b)

$$[\sigma_s, \sigma_r] = (B_s, B_t)_{y} = -[A_s, (B_s, B_t)],$$

(3.8c)

and

$$(A_s, A_t)|_{u=0} = 0,$$

(3.9a)

$$(A_s, B_t)|_{u=0} = sA^{s+r-1} + sA^{s+r-2},$$

(3.9b)

$$(B_s, B_t)|_{u=0} = (s-r)\Delta^{s+r-1} + (2y+n)\Delta^{s+r-2} + (y+n)\Delta^{s+r-3}.$$  

(3.9c)

**Proof.** We only prove the equalities (3.8c) and (3.9c). The others can be obtained in a similar manner. Let us take the L.H.S of (3.8c) and establish the R.H.S. For this purpose, L.H.S of (3.8c) can be expressed in the form

$$[\sigma_s, \sigma_r] = (\sigma_s)'[\sigma_r] - (\sigma_r)'[\sigma_s].$$

(3.10)

By taking the Gateaux derivative of $\sigma_s$ in the direction of $\sigma_r$, we get

$$(\sigma_s)'[\sigma_s] = (B_s - [A_s, B_s])'[\sigma_s] = (B_s)'[\sigma_s] - (A_s)'[\sigma_s]B_s$$

$$= -[A_s, (B_s)'[\sigma_s] - B_s]\sigma_s - [A_s, B_s]'[\sigma_s] + [A_s, B_s]'[\sigma_s]A_t$$

$$= -[A_s, B_s]'[\sigma_s] + [A_s, B_s]'[\sigma_s]A_t$$

Similarly, we have

$$(\sigma_r)'[\sigma_s] = (B_r - [A_s, B_s])'[\sigma_s] = (B_r)'[\sigma_s] - (A_s)'[\sigma_s]B_s$$

$$= -[A_s, (B_r)'[\sigma_s] - B_s]\sigma_s + [A_s, B_s]'[\sigma_s] - [A_s, B_s]'[\sigma_s]A_t$$

$$= -[A_s, B_s]'[\sigma_s] + [A_s, B_s]'[\sigma_s]A_t$$

Thus (3.10) becomes,

$$[\sigma_s, \sigma_r] = (B_s)'[\sigma_s] - (B_r)'[\sigma_s] + [A_s, B_s]'[\sigma_s]$$

(3.11)

Observe that

$$\theta([B_m]'[\sigma_n], [B_m]'[\sigma_n]) = \theta([B_m]'[\sigma_n], [B_m]'[\sigma_n]), \quad \forall m, n \in \mathbb{Z}^+$$

is true.

By making use of Jacobi identity

$$[B_s, A_t] = [B_s, [A_t, A_t]] - [B_s, [B_s, A_t]]$$

in (3.11), we get,

$$[\sigma_s, \sigma_r] = (B_s)'[\sigma_s] - (B_r)'[\sigma_s] + [A_s, B_s]'[\sigma_s]$$

(3.12a)

$$[\sigma_s, \sigma_r] = sK_{s+r-1} + sK_{s+r-2},$$

(3.12b)

$$[\sigma_s, \sigma_r] = (s-r)\sigma_{s+r-1} + \sigma_{s+r-2},$$

(3.12c)

where, $s, r \geq 1$ and we set $K_0 = \sigma_0 = 0$.

**Theorem 2.** The flows $K_s$ and $\sigma_r$ form a Lie algebra with structure

$$[K_s, K_t] = 0,$$

$$[K_s, \sigma_r] = sK_{s+r-1} + sK_{s+r-2},$$

$$[\sigma_s, \sigma_r] = (s-r)\sigma_{s+r-1} + \sigma_{s+r-2},$$

and from (3.8b) and (3.9b) and the isospectral zero curvature representation (2.18a), we arrive

$$\theta = \tilde{A}_y - [A_s, \tilde{A}], \quad \tilde{A}|_{u=0} = 0.$$  

(3.13a)

From (3.6), the above equation has only zero solution $\theta = 0$ and $\tilde{A} = 0$. Therefore, (3.12b) is true. Similarly, by taking

$$\omega = [\sigma_s, \sigma_r] - (s-r)\sigma_{s+r-1} + \sigma_{s+r-2},$$

(3.14)

and observing that $\tilde{B}|_{u=0} = 0$ together with (2.19a), (3.8c) and (3.9c), we then have

$$\omega = \tilde{B}_y - [A_s, \tilde{B}], \quad \tilde{B}|_{u=0} = 0.$$  

(3.17)
Hence, we get \( G = 0 \) and \( B = 0 \), which implies that (3.12c) is also correct. Thus, we complete the proof.

4. Conclusion

The study of differential-difference KP equation from various aspects becomes interesting for many researchers. Though many results were available in the literature, in this paper, we further firmly confirmed that the algebraic structures of symmetries underlying the noncommutative version of differential-difference KP equation goes parallel to the commutative version. Finding the quasi-determinant solutions of the non-isospectral flows of noncommutative differential-difference KP equation will be our future task.

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References